Seeing the Difference: Some Research Results in Difference Equations

Gene Quinn

Research in Mathematics

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How is that possible?

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Isn't Mathematics just a static collection of known facts?

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Isn't Mathematics just a static collection of known facts?

The answer is an *emphatic* **NO!**

Research in Mathematics

Mathematics is a constantly evolving and growing body of knowledge.

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So, how is research in mathematics done?

Research in Mathematics

You get the problem in your head,

Research in Mathematics

You get the problem in your head,

then, you walk around all day thinking about it . . .

Research in Mathematics

You get the problem in your head,

then, you walk around all day thinking about it . . .

and feeling very happy. Gerry Ladas

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Research in Mathematics

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If I didn't do research, how could I face my mother?

Gerry Ladas

Research in Mathematics

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Prior to 1993, most of the papers published by the URI research group were in differential equations.

In 1993, the group decided to shift their focus to difference equations.

Research in Mathematics

We didn't know whether we were heading into the Mathematical equivalent of a desert or not . . .

Research in Mathematics

As it turned out, the streets are paved with gold . . .

Research in Mathematics

As it turned out, the streets are paved with gold . . .

You just have to be strong enough to pick it up.

Ed Grove

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Research in Mathematics

Your 5-Year Research Plan

Presented by Jim Yorke

11th International Conference on Difference Equations

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Opening sentence:

I sincerely hope you haven't got one . . .

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If you visit Gerry Ladas' webpage it says:

Area of Research at the Present Time,

which suggests the possibility of another switch.

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If you visit Gerry Ladas' webpage it says:

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which suggests the possibility of another switch.

If I wasn't doing difference equations, I'd probably be working in combinatorics

Gerry Ladas

Research in Mathematics

If you are thinking of switching specialties, you would have to plan on spending a year or so reading the literature in your new area

Orlando Merino

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Research in Mathematics

At the University of Rhode Island, there is a very active group doing research in difference equations.

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At the University of Rhode Island, there is a very active group doing research in difference equations.

A key ingredient in sustaining the level of research is a graduate course in difference equations research runs every semester.

Research in Mathematics

It's very unpredictable.

The way it works is, somebody gets and idea, and we try to pick up the ball and run with it.

Ed Janowski

Research in Mathematics

But you've already taken this course six times . . .

Staff member, URI registrar's office

Research Area

AMS Subject Classification 39A11

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Stability and asymptotics of difference equations;

oscillatory and periodic solutions, etc.

Difference Equations

A **difference equation** is an equation of the form

$$x_{n+1} \;=\; f(x_n), \quad n = 0, 1, \ldots$$

Difference Equations

A **difference equation** is an equation of the form

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots$$

A **solution** of a difference equation is a **sequence**

$$\{x_n\}_{n=0}^{\infty}$$

that satisfies the difference equation.

Difference Equations

The combination of a difference equation

$$x_{n+1} \;=\; f(x_n), \quad n = 0, 1, \ldots$$

and an **initial condition** x_0 uniquely determines a solution.

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$$x_{n+1} \;=\; f(x_n), \quad n = 0, 1, \ldots$$

and an **initial condition** x_0 uniquely determines a solution. The solution is the **sequence**

$$\{x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots\}$$

Historical Background

The study of systems of this type was pioneered by the French Mathematicians Pierre Fatou and Gaston Julia in the first quarter of the twentieth century.

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They explored maps of the form

$$z_{n+1} = \frac{\alpha + \beta z}{\gamma + \delta z}, \quad z \in \mathbb{C}$$

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In 1975 Li and Yorke published their celebrated

Period Three Implies Chaos

paper (American Mathematical Monthly 82(1975), 985-992).

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Period Three Implies Chaos

paper (American Mathematical Monthly 82(1975), 985-992).

This paper sparked a great deal of interest in dynamical systems and chaos.

39A11 People

July 26, 2006 ICDEA 2006 Kyoto University

The Mandelbrot Set

Around the time of Li and Yorke's paper, Benoit Mandelbrot was producing spectacular computer graphics based on the work of Julia.

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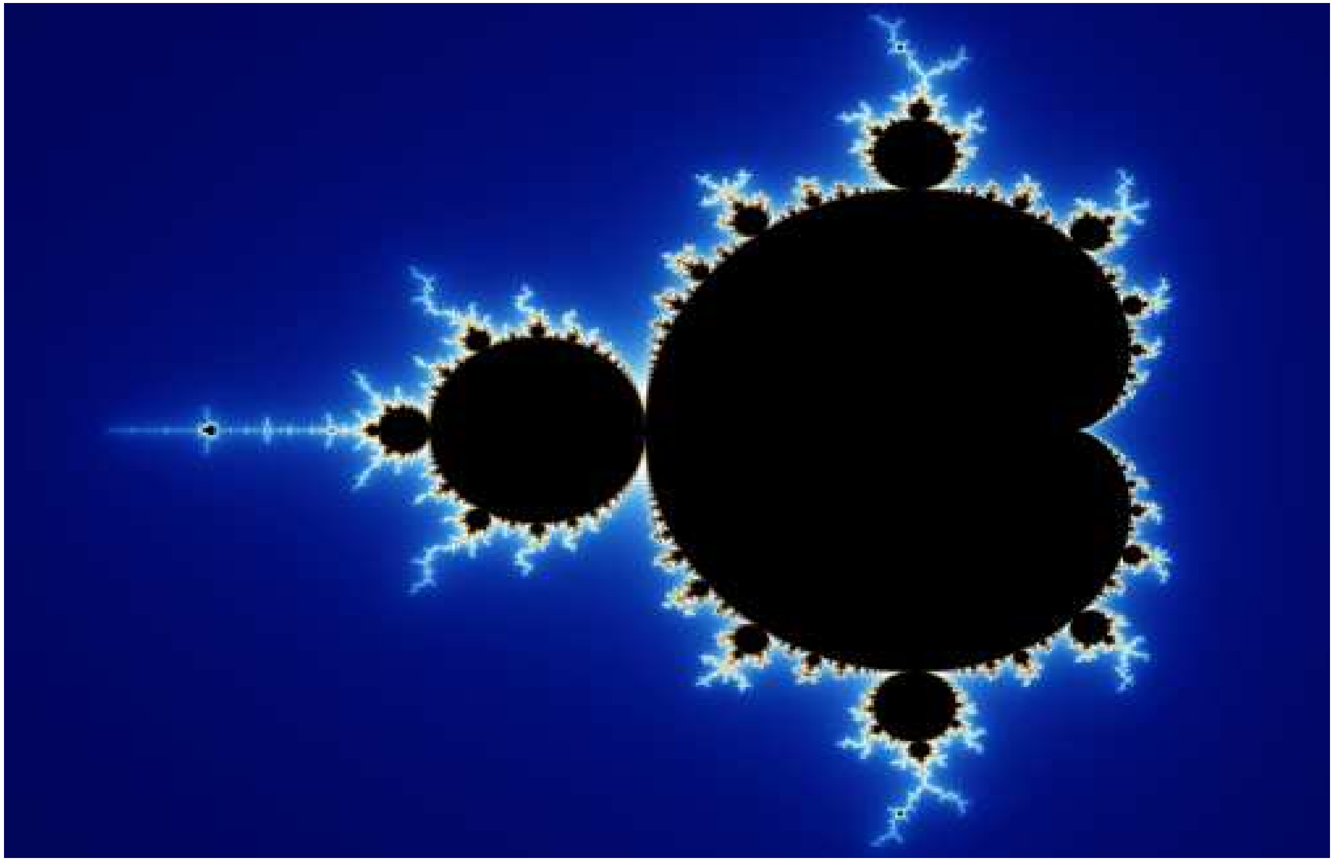
Around the time of Li and Yorke's paper, Benoit Mandelbrot was producing spectacular computer graphics based on the work of Julia.

The best known of Mandelbrot's results investigates the difference equation

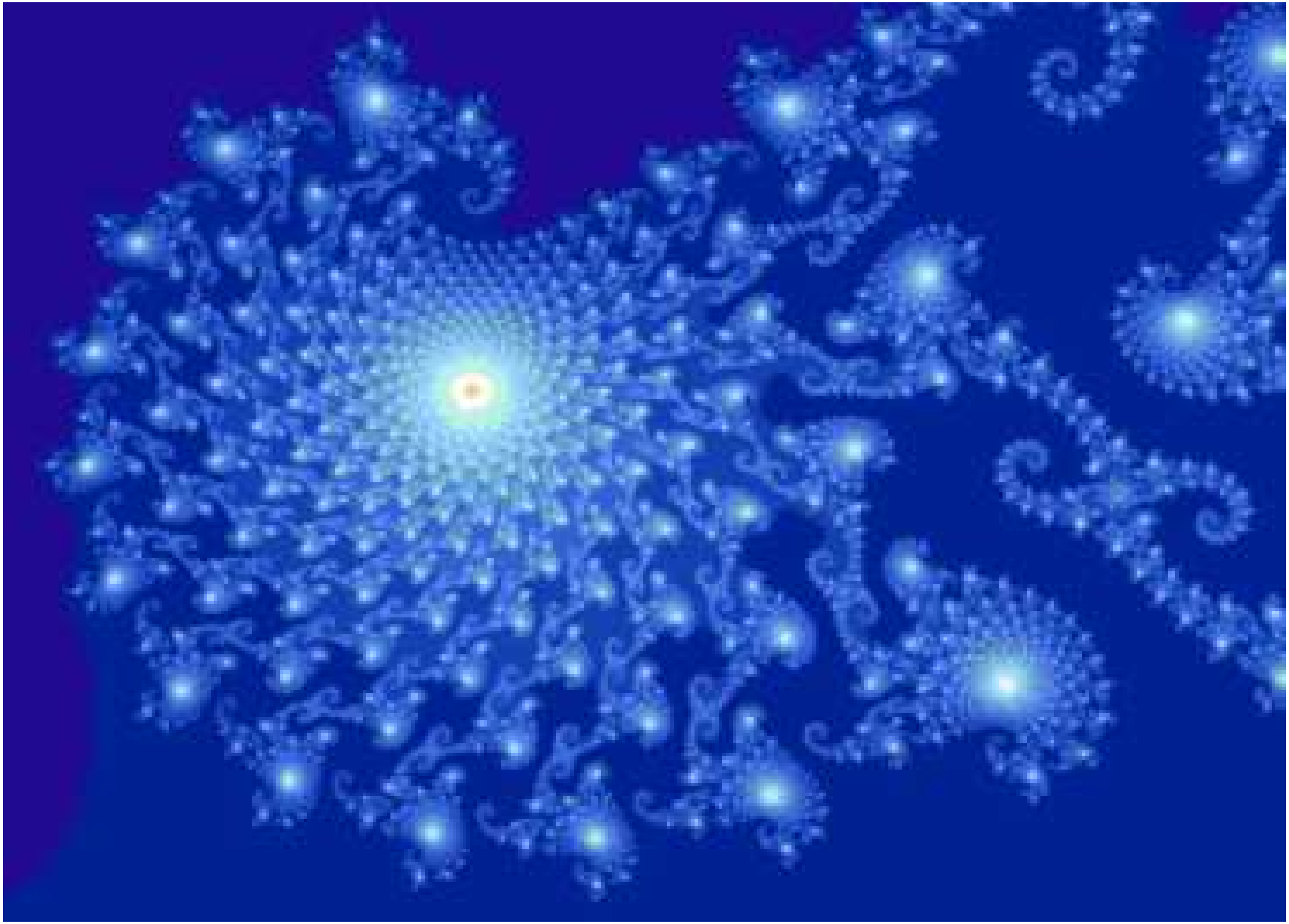
$$z_{n+1} = z_n^2 + c, \quad c \in \mathbb{C}$$

The Mandelbrot Set



$$z_{n+1} \ = \ z_n^2 + c, \quad z_0 = 0, \quad c \in \mathbb{C}$$

The Mandelbrot Set



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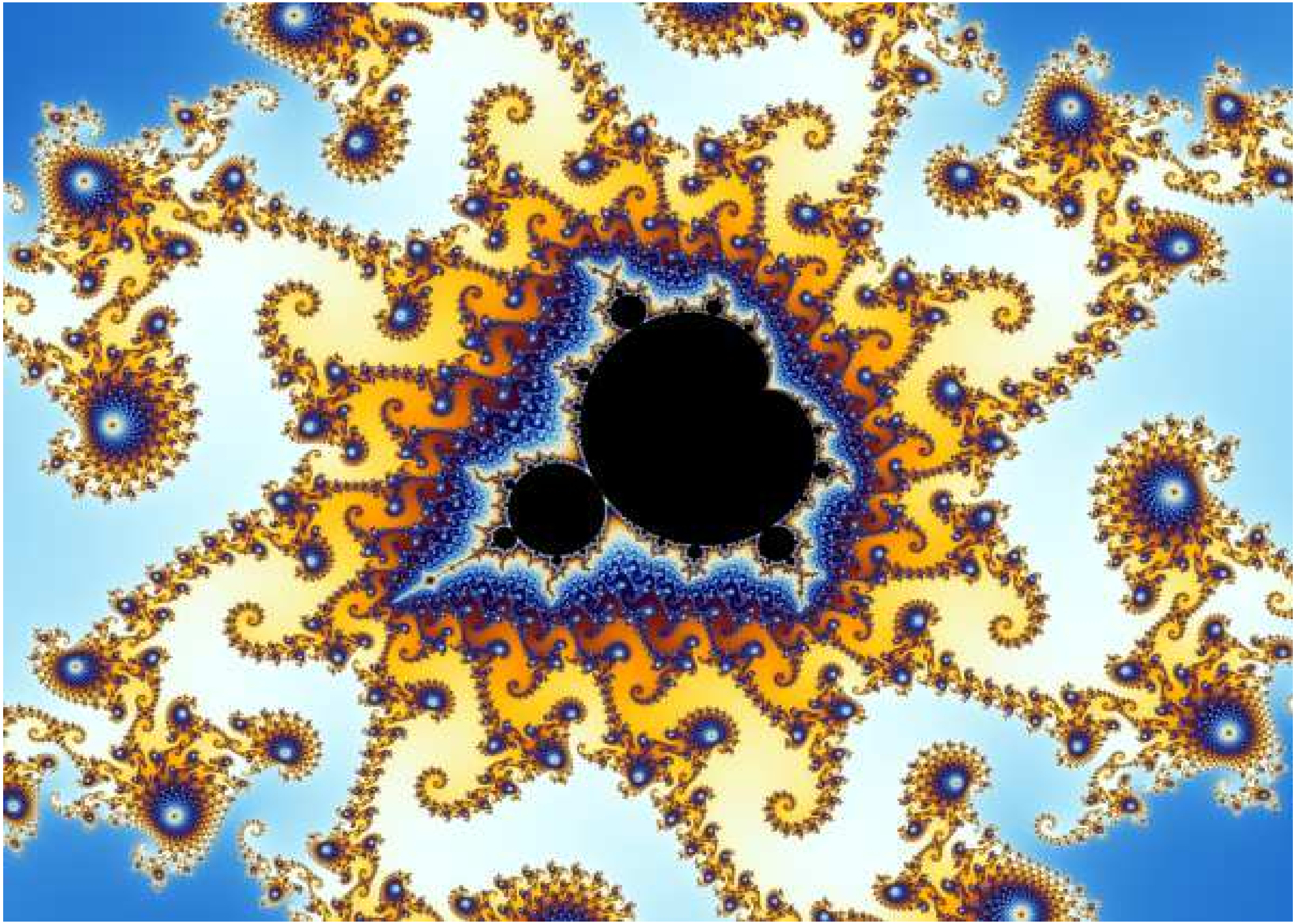

Rational Difference Equations

The recent focus of the URI research group has been *rational* difference equations.

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The recent focus of the URI research group has been *rational* difference equations.

The following is a *first order* rational difference equation:

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + B x_n}, \quad n = 0, 1, \ldots$$

Rational Difference Equations

The following are second and third order rational difference equations:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}, \qquad n = 0, 1, \ldots$$

Rational Difference Equations

The following are second and third order rational difference equations:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}}, \; n = 0, 1, \ldots$$

Difference Equations

The following questions regarding solution sequences are of interest:

- What happens as $n \to \infty$?

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The following questions regarding solution sequences are of interest:

- What happens as $n \to \infty$?
- Does the sequence approach a limit?
- Does the sequence remain bounded?
- Does the sequence become periodic?

Rational Difference Equations

The following third order rational difference equation exhibits all of these behaviors:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n}, \ \ n = 0, 1, \ldots$$

Rational Difference Equations

The following third order rational difference equation exhibits all of these behaviors:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n}, \; n = 0, 1, \ldots$$

As it turns out, the behavior of solutions is almost entirely determined by the parameters γ, β, δ and A.

Rational Difference Equations

When $\gamma < \beta + \delta + A$, every solution of the third order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n}, \; n = 0, 1, \ldots$$

converges to an equilibrium or fixed point $\overline{x}$

Rational Difference Equations

An *equilibrium* or *fixed point* $\overline{x}$ satisfies the equation

$$\overline{x} = \frac{\alpha + \beta\overline{x} + \gamma\overline{x} + \delta\overline{x}}{A + B\overline{x}}$$

Rational Difference Equations

An *equilibrium* or *fixed point* $\overline{x}$ satisfies the equation

$$\overline{x} = \frac{\alpha + \beta\overline{x} + \gamma\overline{x} + \delta\overline{x}}{A + B\overline{x}}$$

It can be shown that if a solution sequence converges to a value, that value must be a fixed point.

Rational Difference Equations

When $\gamma = \beta + \delta + A$, every solution of the third order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n}, \;\; n = 0, 1, \ldots$$

converges to a periodic solution with period 2,

$$\phi, \psi, \phi, \psi, \ldots$$

Rational Difference Equations

When $\gamma > \beta + \delta + A$, the third order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n}, \ n = 0, 1, \ldots$$

has unbounded solutions.

Periodic Solutions

Some difference equations have *every* solution periodic with the same period.

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Some difference equations have *every* solution periodic with the same period.

$$x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, \ldots$$

has every solution periodic with period 2.

Periodic Solutions

It's not as obvious as the previous example, but

$$x_{n+1} = \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots$$

has every solution periodic with period 6.

Boundedness

Some rational difference equations have the property that every solution is bounded.

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Some rational difference equations have the property that every solution is bounded.

Many Mathematical biologists consider this property an absolute must for a population model.

Rational Difference Equations

As it turns out, a rational difference equation that has all possible terms, such as

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}, \; n = 0, 1, \ldots$$

with nonnegative parameters and initial conditions also has every solution bounded.

Proof of Boundedness

Consider the finite set of nonnegative real numbers:

$$\{\alpha, \beta, \gamma, \delta\}$$

Proof of Boundedness

Consider the finite set of nonnegative real numbers:

$$\{\alpha, \beta, \gamma, \delta\}$$

This set has a maximum, call it M.

Proof of Boundedness

Likewise, the set of parameters from the denominator,

$$\{A, B, C, D\}$$

has a minimum, call it m.

Rational Difference Equations

Regardless of the values of the x's, if I replace the parameters in the numerator with their maximum, M, I can write the inequalities

$$
\begin{aligned}
& \alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2} \\
\leq\ & M + M x_n + M x_{n-1} + M x_{n-2} \\
=\ & M\left(1 + x_n + x_{n-1} + x_{n-2}\right)
\end{aligned}
$$

Rational Difference Equations

Now replace the parameters in the denominator with their minimum, m, and write the inequalities

$$\begin{aligned} & A + Bx_n + Cx_{n-1} + Dx_{n-2} \\ \geq \; & m + mx_n + mx_{n-1} + mx_{n-2} \\ = \; & m(1 + x_n + x_{n-1} + x_{n-2}) \end{aligned}$$

Rational Difference Equations

We made the numerator larger and the denominator smaller, so we can write

$$\begin{aligned} x_{n+1} \;&=\; \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}} \\ &\leq \frac{M(1 + x_n + x_{n-1} + x_{n-2})}{m(1 + x_n + x_{n-1} + x_{n-2})} \;=\; \frac{M}{m} \end{aligned}$$

Applications

The most famous scientist at URI is probably Saul Salia, who models marine populations. He mostly uses differential equations.

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The most famous scientist at URI is probably Saul Salia, who models marine populations. He mostly uses differential equations.

He once told me he thought difference equations were much more appropriate, but he used differential equations because they are much better understood.

Ed Grove

Some Quotes

Now, we all know the reason we do this [research] is because we enjoy doing it.

Some Quotes

Now, we all know the reason we do this [research] is because we enjoy doing it.

But, when people ask you why you do this, they won't believe that.

Ed Grove

Some Quotes

I don't study these equations to become famous as the person who discovered something.

I don't care about that.

Some Quotes

I don't study these equations to become famous as the person who discovered something.

I don't care about that.

I study them because they are there and they have to be investigated.

Gerry Ladas

Some Quotes

We are desparately trying to discover results by any means we can.

Gerry Ladas

Some Quotes

If Grove hadn't been hit by that car, we would have proved that [theorem] last summer

Gerry Ladas

Some Quotes

Why aren't these equations in every high school Mathematics book?

Some Quotes

Why aren't these equations in every high school Mathematics book?

In 50 years, they will be.

Gerry Ladas

The End

Thank You!