Theorem:(4.6.3) Given $f : A \to \mathbb{R}$ and a limit point c of A, then

$$\lim_{x\to c} f(x) = L \quad \text{if and only if} \quad \lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)$$

That is, the functional limit $\lim_{x\to c}$ exists if and only if the left and right hand functional limits both exist, and are equal.

Proof. ($\Rightarrow$) First assume

$$\lim_{x\to c} f(x) = L$$

and suppose $\epsilon > 0$ is given. Then there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad |x - c| < \delta$$

Now $|x - c| < \delta$ is equivalent to the double inequality $-\delta < x - c < \delta$, so

$$\text{if} \quad 0 < x - c < \delta \quad \text{then} \quad |x - c| < \delta \quad \text{and therefore} \quad |f(x) - L| < \epsilon$$

which establishes that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < x - c < \delta \quad \text{so} \quad \lim_{x\to c^+} f(x) = L$$

The other part of the double inequality implies that

$$\text{if} \quad -\delta < x-c < 0 \Rightarrow 0 < c-x < \delta \quad \text{then} \quad |x-c| < \delta \quad \text{which implies} \quad |f(x)-L| < \epsilon$$

and we have established that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < c - x < \delta \quad \text{so} \quad \lim_{x\to c^-} f(x) = L$$

so

$$\lim_{x\to c} f(x) = L \quad \text{implies} \quad \lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)$$

($\Leftarrow$) Now suppose $\lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)$. Then by definition, given $\epsilon > 0$, there exists a

$$\delta_1 \quad \text{such that} \quad |f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < x - c < \delta_1$$

and a

$$\delta_2 \quad \text{such that} \quad |f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < c - x < \delta_2$$

Let δ be the smaller of δ_1 and δ_2. Then since

$$0 < x - c < \delta \quad \text{or} \quad 0 < c - x < \delta \quad \text{is equivalent to} \quad |x - c| < \delta$$

we have that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad |x - c| < \delta$$

which establishes that $\lim_{x\to c} f(x) = L$. $\square$