Theorem:(4.6.8) A function that is continuous on a compact set K is uniformly continuous on K.

Proof. Suppose $\epsilon > 0$ is given. The continuity of f on K guarantees that for each $x \in K$, there exists a $\delta_x > 0$, which depends on x, such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad \text{whenever} \quad |x - y| < \delta_x \quad \text{and} \quad y \in K$$

Now form an open cover for K by taking the union of the $(\delta_x/2)$-neighborhood of each $x \in K$,

$$O = \bigcup_{x \in K} V_{\frac{1}{2}\delta_x}(x)$$

By hypothesis K is compact, so there exists a finite subcover corresponding to a finite set of points $\{x_1, \ldots, x_n\}$ in K,

$$K \subseteq V_{\frac{1}{2}\delta_{x_1}}(x_1) \cup V_{\frac{1}{2}\delta_{x_2}}(x_2) \cup \cdots \cup V_{\frac{1}{2}\delta_{x_n}}(x_n)$$

Because n is finite, the set of δ_{x_i} values has a minimum, which must be positive, so let

$$\delta \ = \ \min\left\{\frac{1}{2}\delta_{x_1}, \frac{1}{2}\delta_{x_2}, \ldots, \frac{1}{2}\delta_{x_n}\right\}$$

Now suppose $x, y \in K$ are arbitrarily chosen elements of K with $|x - y| < \delta$. Because the $(\delta_{x_i}/2)$-neighborhoods form an open cover for K, there is some x_i such that

$$|x - x_i| < \frac{1}{2}\delta_{x_i} < \delta_{x_i} \quad \text{which implies that} \quad |f(x) - f(x_i)| < \frac{\epsilon}{2}$$

Because $|x - y| < \delta$, by the triangle inequality

$$|y - x_i| \ \leq \ |y - x| + |x - x_i| < \delta + \frac{1}{2}\delta_{x_i} < \delta_{x_i}$$

since δ is the smallest value of $\delta_{x_i}/2$, but the δ_{x_i} values were chosen so that

$$|f(y) - f(x_i)| < \frac{\epsilon}{2} \quad \text{whenever} \quad |y - x_i| < \delta_{x_i}$$

so, using the triangle inequality again, we obtain

$$|f(y) - f(x)| \leq |f(y) - f(x_i)| + |f(x_i) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so we can say that there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \epsilon \quad \text{whenever} \quad |x - y| < \delta \quad \text{and} \quad x, y \in K$$

which estabishes that f is uniformly continuous on K. □