1. Part a)

We want to show that for any $s \in [0, 2]$, there are two elements x and y in the Cantor set C with the property that $x + y = s$. This means that the set of all pairwise sums of elements of C is the interval $[0, 2]$.

$$\{x + y \mid x, y \in C\} = [0, 2]$$

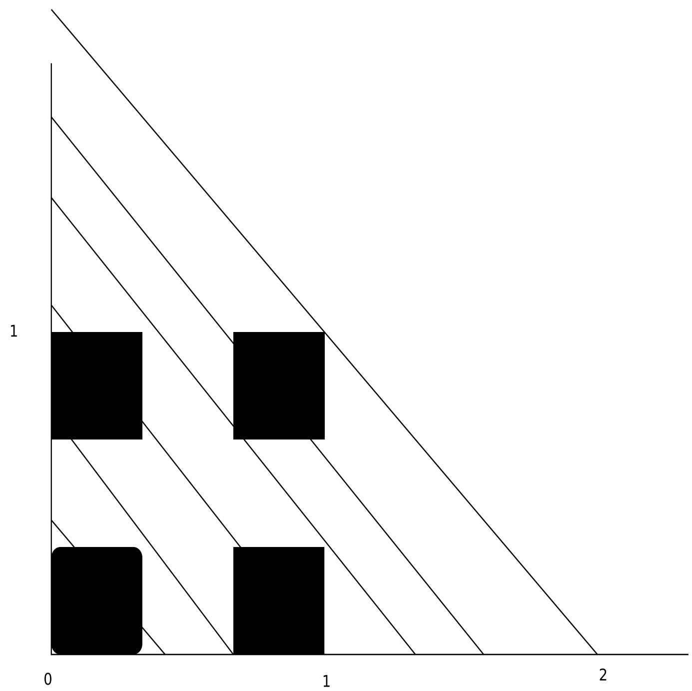
Recall that C_1 is the interval $[0, 1]$ with the middle third removed:

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Let $s \in [0, 2]$ be an arbitrary real number between 0 and 2, inclusive.

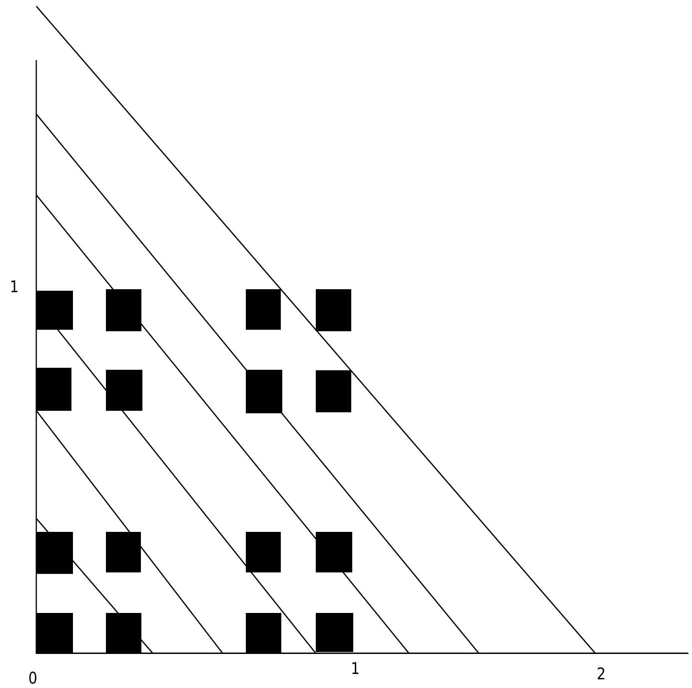
We begin by arguing that there are two elements x_1 and y_1 with $x_1 + y_1 = s$.

Solutions of the equation $x+y = s$ lie on a straight line with equation $y = -x + s$, which is a line with slope -1 and y-intercept s (the x-intercept is also s). If we graph members of this family of lines with various values of s on the same set of axes with C_1, we get the following picture:



A bit of thought and possibly some trial-and-error computations should convince you that it is impossible to draw a line with slope -1 and y-intercept ≤ 2 that does not intersect C_1 (the four shaded squares). Consequently, for *any* given $s \in [0, 2]$, we can find at least one pair of values $x_1, y_1 \in C_1$ such that the point (x_1, y_1) lies on the line $y = -x + s$, which is equivalent to saying a pair $(x_1, y_1) \in C_1$ such that $x_1 + y_1 = s$.

Now consider what happens with C_2. Each of the four squares in C_1 ends up subdivided in exactly the same way $[0,1]$ was subdivided to produce C_1: So we can construct an induction argument in the following way. We know that for any $s \in [0,2]$, the line $x + y = s$ intersects one of the four square shaded region



Now we can construct an induction argument in the following way. We know that for any $s \in [0,2]$, the line $x + y = s$ intersects one of the four square shaded regions in C_1. Now suppose the line $s = x + y$ intersects C_n at the point (x_n, y_n), which means it touches at least one of the shaded squares in C_n. Moving to the $n + 1^{st}$, we remove the middle third of the shaded region, but this leaves a smaller image identical to C_1, so by the same argument as we used for C_1, the line $x + y = s$ must intersect C_{n+1} at some point (x_{n+1}, y_{n+1}).

2. Part b)

By the induction argument presented in part a), we produced a sequence of pairs x_n, y_n with $x_n + y_n = s$. Because we can't be sure which shaded square of C_n the line $x + y = s$ intersects, we can't say that x_n converges. However, it is bounded, so the Bolzano-Weierstrass Theorem guarantees the existence of a convergent subsequence x_{n_i}. Suppose $\lim x_{n_i} = x$.

Because $x_n + y_n = s$ for every n, $y_n = s - x_n$, so $\lim y_n = y = s - x$. It remains to show that x and y are in

$$C = \bigcap_{n=1}^{\infty} C_n$$

We can say that all terms of (x_{n_i}) are in C_1, and because C_1 is closed, so is the $\lim x_{n_i} = x$. Next, all of (x_{n_i}) except x_{n_1} is in C_2, which is closed, so $x \in C_2$, all but x_{n_1} and x_{n_2} are in C_3, which is also closed, and so on, so that for any $k \in \mathbb{N}$, all terms after x_{n_k} are in C_k, a closed set, so $\lim x_n = x$ is in C_k as well. Because k was arbitrary, this means x belongs to C_k for every $k \in \mathbb{N}$, which means that

$$x \in \bigcap_{n=1}^{\infty} C_n = C$$

A similar argument works for y, so $x + y = s$ with $x, y \in C$.