Hint 1. To compute

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \frac{\sin nx}{n}$$

note that for any n and x,

$$-1 \leq \sin nx \leq 1 \quad \text{which implies that} \quad -\frac{1}{n} \leq \frac{\sin nx}{n} \leq \frac{1}{n}$$

For uniform convergence, given $\epsilon > 0$, try to choose N in the definition of the limit so that it does not depend on x.

Hint 2. How does the picture change for if we replace n in the denominator by $\sqrt{n}$?

Hint 3. What is the value of $h_n'(x)$ if $x = \pi/2$? Are there other values of x that have this property?

Hint 4. Note that, for $x \in [0,1]$,

$$\frac{x^n}{n} \leq \frac{1}{n}$$

Hint 5. Compare the functions

$$g_n(x) = \frac{nx + x^2}{2n}$$

to Example 6.2.2 (i).

Hint 6. For

$$f_n(x) = \frac{nx^2 + 1}{2n + x}$$

consider the maximum and minimum values of

$$|f_n'(x) - x| = \left| \frac{-nx^2 - x^3 + 1}{4n^2 + 4nx + x^2} \right|$$

on the set $[-M, M]$. Can these be written in a way that does not depend on x?

Hint 7. The result of applying the mean value theorem to $f_n - f_m$ is

$$(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0)) = (fn'(\alpha) - f_m'(\alpha))(b - a)$$