Hint 1. To compute

$$\lim_{n\to\infty} \frac{\sin nx}{n}$$

note that for any n and x,

$$-1 \le \sin nx \le 1 \quad \text{which implies that} \quad -\frac{1}{n} \le \frac{\sin nx}{n} \le \frac{1}{n}$$

Hint 2. A sequence $(f_n(x))$ with $f_n : A \to \mathbb{R}$ converges uniformly to a limit function $f : A \to \mathbb{R}$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{whenever} \quad n \ge N, \quad \text{for every} \;\; x \in A$$

One way a sequence $(f_n(x))$ with $f_n : A \to \mathbb{R}$ *fails to converge uniformly* to a limit function $f : A \to \mathbb{R}$ is if for *any* $\epsilon > 0$, for *every* $N \in \mathbb{N}$, there exists an $x \in A$ that makes

$$|f_n(x) - f(x)| \ge \epsilon \quad \text{regardless of the value of} \;\; n$$

For example, suppose $f_n : (0, \infty) \to \mathbb{R}$ is defined by

$$f_n(x) = \frac{nx}{1 + nx^2} \quad \text{then} \quad \lim_{n\to\infty} f_n(x) = \frac{1}{x}$$

and

$$|f_n(x) - f(x)| = \left| \frac{nx}{1 + nx^2} - \frac{1}{x} \right| = \frac{1}{x + nx^3}$$

For any given $\epsilon > 0$ and for any n, the rightmost quantity can be made greater than ϵ by choosing x sufficiently close to zero:

$$\epsilon \le \frac{1}{x + nx^3} < \frac{1}{nx^3} \quad \Leftrightarrow \quad x < \frac{1}{\sqrt[3]{n\epsilon}}$$

Hint 3. Note that

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if} \;\; 0 \le x < 1 \\ 1 & \text{if} \;\; x = 1 \\ \infty & \text{if} \;\; x > 1 \end{cases}$$

Hint 4. By setting the derivative equal to zero and solving for x, the maximum absolute value assumed by

$$|f_n(x)| = \left|\frac{x}{1+nx^2}\right| \quad \text{is} \quad \frac{1}{2\sqrt{n}}$$

so

$$-\frac{1}{2\sqrt{n}} < f_n(x) < \frac{1}{2\sqrt{n}}$$

Hint 5. Note that if

$$f_n(x) = \begin{cases} 1 & \text{if } x \le -1/n \\ -nx & \text{if } -1/n < x \le 0 \\ nx & \text{if } 0 < x < 1/n \\ 1 & \text{if } x \ge 1/n \end{cases}$$

then as $n \to \infty$, the interval $(-1/n, 1/n)$ tends to zero.

Hint 6. Given $f_n : A \to \mathbb{R}$, consider

$$\lim_{n\to\infty} f_n(x)$$

as a candidate for the limit function.

Hint 7. To establish the uniform continuity of a the limit function f of a sequence of functions $f_n : A \to \mathbb{R}$ that converges uniformly, consider the identity

$$|f(x) - f(y)| = |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)|$$

as a starting point for an "$\epsilon/3$" proof. as a candidate for the limit function.

Hint 8. If g is a continuous function defined on a compact set K that is never zero, then

$$\frac{1}{g} \quad \text{is bounded on} \quad K$$

Hint 9. If $f_n(x) = f(x + 1/n)$ and the limit function f is uniformly continuous, consider the implications of the definition of uniform continuity, namely for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(y) - f(x)| = |f(x + 1/n) - f(x)| < \epsilon \quad \text{whenever} \quad \left| x + \frac{1}{n} - x \right| < \delta$$

Also consider what happens if f is not uniformly continuous, for example, x^2.

Hint 10. With regard to whether the product $(f_n g_n)$ converges uniformly or not, consider the case where one of the limit functions is unbounded and the case where both are bounded.

Hint 11. If $f_n \to f$ pointwise on a compact set K and for $\epsilon > 0$, if

$$K_n = \{x \in K \ : \ g_n(x) \geq \epsilon\} \quad \text{where} \quad g_n = f_n - f$$

you can argue that the K_n are closed and bounded to show compactness.

Hint 12. In the construction of the Cantor Function, note that when $m < n$,

$$|f_n(x) - f_m(x)| \leq \frac{1}{2^m}$$

Theorem 6.2.6 applies, as well as Exercise 6.2.8e. What values are assigned to 0 and 1 by each f_n?

Hint 13. If $A = \{x_1, x_2, \ldots\}$ is a countable set and f_n a bounded function on A, the sequence

$$(f_n(x_1)) = (f_1(x_1), f_2(x_1), \ldots)$$

is a bounded sequence of real numbers, and the Bolzano-Weirstrass Theorem implies that there is a convergent subsequence (using the author's notation)

$$(f_{n_k}(x_1)) = f_{1,k}(x_1) = (f_{1,1}(x_1), f_{1,2}(x_1), f_{1,3}(x_1), \ldots)$$

Now consider that, if we apply each function in the sequence to x_2 instead of x_1, the result is also a bounded sequence

$$(f_{1,k}(x_2)) = (f_{1,1}(x_2), f_{1,2}(x_2), f_{1,3}(x_2), \ldots)$$

which will contain a convergent subsequence

$$(f_{2,k}(x_2)) = (f_{2,1}(x_2), f_{2,2}(x_2), f_{2,3}(x_2), \ldots)$$

and this process can be continued by substituting x_3 for x_2 in this subsequence, obtaining another convergent subsequence, substituting x_4 for x_3 in it, and so on, until we have constructed a family of subsequences $f_{m,k}$.

Now consider the properties of the sequence

$$f_{n,k} = (f_{1,1}, f_{2,2}, f_{3,3}, \ldots)$$

for any $x \in A$.

Hint 14. Keep in mind that a sequence of functions that is continuous on $[0, 1]$, or any other compact set, is uniformly continuous on that set.

Hint 15. For the Arzela-Ascoli theorem, consider

$$|g_x(x) - g_t(x)| = |g_s(x) - g_s(r_i) + g_s(r_i) - g_t(r_i) + g_t(r_i) - g_t(x)|$$

and apply the triangle inequality (in a somewhat more general form) to produce an "$\epsilon/3$" argument.