Period-Two Convergence and Existence of Unbounded Solutions in Third-Order Rational Difference Equations

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Third-Order Rational Equations

A *third-order rational difference equation* is a recursive sequence of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots \tag{1}$$

Third-Order Rational Equations

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots$$

The goal of our present research on rational difference equations is to describe the character of solutions of this equation for all possible parameter values and initial conditions.

We are especially interested in the asymptotic behavior of solutions as $n \to \infty$.

Nonnegativity Conditions

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots$$

We will consider only the case where all parameters and initial conditions are nonnegative:

$$\alpha, \beta, \gamma, \delta, A, B, C, D, x_{-2}, x_{-1}, x_0 \in [0, \infty)$$

Nonnegativity Conditions

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots$$

We further assume that the parameters and initial conditions are chosen so that the denominator of the recursive relation is always positive:

$$A + B x_n + C x_{n-1} + D x_{n-2} \; > \; 0, \quad n = 0, 1, \ldots$$

Asymptotic Behavior of Solutions

The asymptotic behavior of solutions of a third-order rational difference equation generally depend on both the parameters

$$\alpha, \beta, \gamma, \delta, A, B, C, D$$

and the initial conditions

$$x_{-2}, x_{-1}, x_0$$

Asymptotic Behavior of Solutions

Solutions of a third-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots$$

may exhibit one or more of the following characteristics:

- The solution converges to an equilibrium $\overline{x}$.
- The solution converges to a periodic solution.
- The solution sequence is unbounded.
- The solution sequence is bounded but does not converge to an equilibrium.
- Every solution is periodic with the same period.

Convergence to an Equilibrium

A solution of a third-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots$$

is said to **converge to an equilibrium** $\overline{x}$ if the solution sequence converges to

$$\ldots, \ \overline{x}, \ \overline{x}, \ \overline{x}, \ \ldots$$

Convergence to a Period-2 Solution

A solution of a third-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots$$

is said to **converge to a period-two solution** if the solution converges to a sequence of the form

$$\ldots,\ \phi,\ \psi,\ \phi,\ \psi,\ \ldots$$

To distinguish the degenerate case where $\phi = \psi$ we refer to period-two solutions with $\phi \neq \psi$ as **prime period-two solutions**.

Trichotomy Character of Solutions

Solutions of a rational difference equation may exhibit the following **trichotomy character**:

$\gamma < \beta + \delta + A \quad \Rightarrow$ Every solution converges to $\overline{x}$

$\gamma = \beta + \delta + A \quad \Rightarrow$ Every solution converges to period-2

$\gamma > \beta + \delta + A \quad \Rightarrow$ Unbounded solutions exist

Known Period-2 Trichotomies

C.H.Gibbons, M.R.S.Kulenovic, G.Ladas, and H.Voulov showed that solutions of the second-order rational equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + x_n}, \quad n = 0, 1, \ldots$$

exhibit trichotomy character in the following form:

$\gamma < \beta + A \quad \Rightarrow$ Every solution converges to $\overline{x}$

$\gamma = \beta + A \quad \Rightarrow$ Every solution converges to period-2

$\gamma > \beta + A \quad \Rightarrow$ Unbounded solutions exist

Known Period-2 Trichotomies

E.Camouzis, G.Ladas, and H.Voulov showed that solutions of the third-order rational equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + x_{n-2}}, \quad n = 0, 1, \ldots$$

exhibit trichotomy character in the following form:

$\gamma < \delta + A \quad \Rightarrow$ Every solution converges to $\overline{x}$

$\gamma = \delta + A \quad \Rightarrow$ Every solution converges to period-2

$\gamma > \delta + A \quad \Rightarrow$ Unbounded solutions exist

Known Period-2 Trichotomies

Finally, E.Chatterjee, E.Grove, Y.Kostrov, and G.Ladas, showed that solutions of the third-order rational equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + Bx_n + x_{n-2}}, \quad n = 0, 1, \ldots$$

exhibit trichotomy character in the following form:

$\gamma < A \quad \Rightarrow$ Every solution converges to $\overline{x}$

$\gamma = A \quad \Rightarrow$ Every solution converges to period-2

$\gamma > A \quad \Rightarrow$ Unbounded solutions exist

Known Period-2 Trichotomies

The three equations

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + x_n}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + x_{n-2}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + B x_n + x_{n-2}}, \quad n = 0, 1, \ldots$$

are the only rational equations known to possess the period-2 trichotomy character.

Partial Trichotomies

E.Camouzis, G.Ladas, and E.P.Quinn showed that solutions of the third-order rational equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \ldots$$

exhibit part of the trichotomy character:

$\gamma = \beta + \delta + A \quad \Rightarrow$ Every solution converges to period-2

$\gamma > \beta + \delta + A \quad \Rightarrow$ Unbounded solutions exist

(period-two convergence requires the additional hypothesis that $\beta + A > 0$).

Existence of Unbounded Solutions

E.Camouzis, G.Ladas, and E.P.Quinn have shown that the third-order rational equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + x_{n-1}}, \quad n = 0, 1, \ldots$$

with

$$\delta, B \in (0, \infty) \quad \text{and} \quad \alpha, \beta, \gamma, A \in [0, \infty)$$

has unbounded solutions when

$$\delta > A + \gamma B + \frac{\beta}{B}$$

Existence of Unbounded Solutions

The proof of this statement proceeds by establishing a very large set of initial conditions that guarantee an unbounded solution when the stated conditions on the parameters are met, namely,

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + x_{n-1}}, \quad n = 0, 1, \ldots$$

with

$$\delta, B \in (0, \infty) \quad \text{and} \quad \alpha, \beta, \gamma, A \in [0, \infty)$$

and

$$\delta > A + \gamma B + \frac{\beta}{B}$$

Existence of Unbounded Solutions

Theorem: Assume that

$$\delta, B \in (0, \infty) \quad \text{and} \quad \alpha, \beta, \gamma, A \in [0, \infty) \ .$$

Then the rational equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + x_{n-1}}, \quad n = 0, 1, \ldots$$

has unbounded solutions in some range of its parameters, and in particular when

$$\delta > A + \gamma B + \frac{\beta}{B}$$

Existence of Unbounded Solutions

Proof: Choose positive numbers m and ϵ such that

$$m \quad \in \quad \left(0, \delta - A - \gamma B + \frac{\beta}{B}\right)$$

and

$$\epsilon \quad \in \quad \left(0, \frac{m}{1+B}\right)$$

Existence of Unbounded Solutions

Now let

$$K = \frac{1}{\epsilon}\left[\alpha + \beta\left(\epsilon + \frac{\beta}{B}\right) + \delta\left(\epsilon + \gamma\right)\right]$$

and

$$L = \frac{1}{\epsilon B}\left[\alpha + \gamma(\epsilon + \gamma) + \delta\left(\epsilon + \frac{\beta}{B}\right)\right]$$

Existence of Unbounded Solutions

Choose initial conditions x_{-2}, x_{-1}, and x_0 as follows:

$$\begin{aligned} x_{-2} &> \max\{K, L\} \\ x_{-1} &\in \left(0, \epsilon + \frac{\beta}{B}\right) \\ x_0 &\in (0, \epsilon + \gamma) \end{aligned}$$

Then we claim that:

$$\begin{aligned} \lim_{n\to\infty} x_{3n} &= \gamma \\ \lim_{n\to\infty} x_{3n+1} &= \infty \\ \lim_{n\to\infty} x_{3n+2} &= \frac{\beta}{B} \end{aligned}$$

Existence of Unbounded Solutions

Under the stated conditions,

$$x_1 = \frac{\alpha + \beta x_0 + \gamma x_{-1} + \delta x_{-2}}{A + B x_0 + x_{-1}}$$

$$> \frac{\alpha + \beta x_0 + \gamma x_{-1}}{A + \gamma B + \frac{\beta}{B} + \epsilon (1 + B)} + \frac{\delta x_{-2}}{A + \gamma B + \frac{\beta}{B} + \epsilon (1 + B)}$$

$$> \frac{\alpha + \beta x_0 + \gamma x_{-1}}{\delta} + \frac{\delta}{A + \gamma B + \frac{\beta}{B} + m} \cdot x_{-2}$$

Existence of Unbounded Solutions

In addition,

$$x_2 = \frac{\alpha + \beta x_1 + \gamma x_0 + \delta x_{-1}}{A + Bx_1 + x_0}$$

$$< \frac{\alpha + \gamma(\epsilon + \gamma) + \delta\left(\epsilon + \frac{\beta}{B}\right) + \beta x_1}{Bx_1}$$

$$< \frac{\alpha + \gamma(\epsilon + \gamma) + \delta\left(\epsilon + \frac{\beta}{B}\right) + \beta L}{BL} = \epsilon + \frac{\beta}{B}$$

Existence of Unbounded Solutions

And,

$$x_3 = \frac{\alpha + \beta x_2 + \gamma x_1 + \delta x_0}{A + B x_2 + x_1}$$

$$< \frac{\alpha + \beta\left(\epsilon + \frac{\beta}{B}\right) + \delta\left(\epsilon + \gamma\right) + \gamma x_1}{x_1}$$

$$< \frac{\alpha + \beta\left(\epsilon + \frac{\beta}{B}\right) + \delta\left(\epsilon + \gamma\right) + \gamma K}{K} = \epsilon + \gamma$$

Existence of Unbounded Solutions

It follows by induction that for $n \geq 0$,

$$x_{3n+1} > \frac{\alpha + \beta x_{3n} + \gamma x_{3n-1}}{\delta} + \frac{\delta}{A + \gamma B + \frac{\beta}{B} + m} \cdot x_{3n-2} \ ,$$

$$x_{3n+2} < \ \epsilon + \frac{\beta}{B}$$

$$x_{3n} < \ \epsilon + \gamma$$

Existence of Unbounded Solutions

By hypothesis, the right hand term of the right side of the inequality

$$x_{3n+1} > \frac{\alpha + \beta x_{3n} + \gamma x_{3n-1}}{\delta} + \frac{\delta}{A + \gamma B + \frac{\beta}{B} + m} \cdot x_{3n-2} \ ,$$

is nonnegative, and the condition that

$$m \ \in \ \left(0, \delta - A - \gamma B - \frac{\beta}{B}\right)$$

ensures that the (constant) multiplier of x_{3n-2} is > 1.

Existence of Unbounded Solutions

This establishes that

$$x_{3n+1} \to \infty \quad \text{as} \quad n \to \infty,$$

while x_{3n+2} and x_{3n} remain bounded by $\epsilon + \frac{\beta}{B}$ and $\epsilon + \gamma$, respectively.

Existence of Unbounded Solutions

Since, as $n \to \infty$,

$$x_{3n+1} \to \infty$$

while x_{3n} and x_{3n+2} remain bounded, we have the following limit:

$$x_{3n+2} = \frac{\alpha + \beta x_{3n+1} + \gamma x_{3n} + \gamma x_{3n-1}}{A + B x_{3n+1} + x_{3n}} \to \frac{\beta}{B}$$

as $n \to \infty$.

Existence of Unbounded Solutions

In similar fashion,

$$x_{3n+3} = \frac{\alpha + \beta x_{3n+2} + \gamma x_{3n+1} + \gamma x_{3n}}{A + B x_{3n+2} + x_{3n+1}} \;\rightarrow\; \gamma$$

as $n \rightarrow \infty$, and the proof is complete.

References

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