This material is background for transformations of random variables.

1. Definitions

1.1. Functions.

A *function*

$$f : X \to Y$$

is a *mapping* that associates with each element of a set X a **unique** element of another set Y.

Recall that the *Cartesian product* $X \times Y$ of two sets X and Y is defined as:

$$X \times Y \;=\; \{(x, y) \,:\, x \in X \quad \text{and} \quad y \in Y\}$$

One can define a function $f : X \to Y$ in a very general and abstract way as a subset of $X \times Y$ with the restriction that each element of X appears in *exactly one* ordered pair.

(This level of abstraction is not usually required, but it clarifies the role of the set Y)

The set X is called the **domain** of f. *Every* element of the domain must be associated with exactly one element of Y.

The set Y usually does not have a name, though some authors call it the **codomain**.

The set of values in Y that f actually maps one or more elements of X to is called the **range** of f, and in set builder notation the definition of the range of f is:

$$\{y \in Y \,:\, (\exists x)[y = f(x)]\}$$

It is not required that f map at least one element of X to every element of Y, but if it does, f is said to be *onto* or *surjective.*

1.2. Images.

It is useful to extend the concept of a function as a mapping to include subsets of the domain as well as individual elements.

If $A \subseteq X$ is some subset of the domain, we define the *image* under f of A to be the set of all elements $y \in Y$ such that $y = f(x)$ for some $x \in A$, and denote it by $f[A]$:

$$f[A] \;=\; \{y \in Y \,:\, (\exists x)[x \in A \quad \text{and} \quad y = f(x)]\}$$

With this notation, the range of X is $f[X]$. The function f is onto if and only if $f[X] = Y$.

A more important notion than the image under f of a subset of X is the idea of an *inverse image*. If $B \subseteq Y$, we define the **inverse image** $f^{-1}[B]$ of B to be the set of elements of X for which $f(x) \in B$:

$$f^{-1}[B] \;=\; \{x \in X \;:\; f(x) \in B\}$$

The function f is onto (surjective) if and only if the inverse image of every nonempty subset B of Y is nonempty.

The notation $f^{-1}[B]$ for the inverse image of B should not be confused with the inverse function f^{-1}. The (possibly empty) inverse image always exists for any subset of B, but we assume nothing about the existence of the function inverse f^{-1}.

1.3. **Function Inverses.** Recall that a function $f : X \to Y$ is called *one-to-one* (or *injective* or *univalent*) if for any distinct elements $x_1, x_2 \in X$,

$$f(x_1) \;=\; f(x_2) \quad \text{only if} \quad x_1 = x_2$$

A function that is both one-to-one and onto is called a *one-to-one correspondence* (or *bijection*). In this case there exists a function $g : Y \to X$ called the *inverse* of f with the property that

$$g(f(x)) \;=\; x \quad \forall x \in X \quad \text{and} \quad f(g(y)) = y \quad \forall y \in Y$$

When the inverse of f exists, it is usually denoted by f^{-1}. The notation for the inverse image of a subset of Y

$$f^{-1}[A], \quad A \subseteq Y$$

is actually ambiguous when $g = f^{-1}$ exists, but it's not problem because in this case, the image of A under f^{-1} and the inverse image of A under f are the same.

1.4. **Properties of Inverse Images.** The inverse image is very well-behaved with respect to the usual set operations of union, intersection, and complimentation. For an arbitrary $B \subseteq Y$ and arbitrary collections $\{B_\lambda\}$ of subsets of Y,

$$f^{-1}\left[\bigcup B_\lambda\right] \;=\; \bigcup f^{-1}\left[B_\lambda\right]$$

$$f^{-1}\left[\bigcap B_\lambda\right] \;=\; \bigcap f^{-1}\left[B_\lambda\right]$$

$$f^{-1}[\sim B] \; = \sim f^{-1}[B] \quad \text{for} \quad B \subset Y$$

where $\sim B$ denotes the compliment of B with respect to Y.

1.5. **Random Variabe Transformations.** In the context of a random variable transformation, the sets X and Y in the definition of the transform function $f : X \to Y$ may be taken respectively to be the support of some random variables X and Y, and we usually consider maps that are onto, that is, $f[X] = Y$.

In this context the following general principle applies.

If f maps some random variable X into another random variable Y, for any subset $B \subseteq Y$, we require that

$$P(y \in B) \; = \; P(x \in f^{-1}[B])$$

That is, for any subset B of the support of Y, *the probability that an observation from the population represented by Y is an element of B must be the same as the probability that an observation from the distribution X belongs to the inverse image under f of B, namely $f^{-1}[B]$.*

1.6. **Some Examples.** Usually we will be interested in subsets of Y that are intervals. For example, suppose X is a normally distributed random variable with mean $\mu = 0$ and variance $\sigma^2 = 1$, denoted by $X \sim N(0, 1)$. Suppose also that we are interested in the random variable $Y = X^2$.

While X can assume any real value, Y assumes only nonnegative values. The map is not one-to-one, and a bit of thought should convince you that, for $B \subseteq Y$ of the form $[0, y]$,

$$f^{-1}[B] \; = \; f^{-1}[[0, y]] \; = \; [-\sqrt{y}, \sqrt{y}]$$

The CDF of Y, $F_Y(y)$ is the probability that an observation falls in $[0, y]$. This should be the same as the probability that an observation of X falls in

$$f^{-1}[[0, y]] \; = \; [-\sqrt{y}, \sqrt{y}]$$

so

$$F_Y(y) \; = \; \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

To obtain the PDF $f_Y(y)$, we can differentiate the CDF with respect to y. Recall from the fundamental theorem of calculus that

$$\frac{d}{dy} \int_a^{u(y)} f(x)\, dx \; = \; f(u(y)) \cdot \frac{du(y)}{dy}$$

so

$$f_Y(y) = \frac{d}{dy} F_Y(y) \; = \; \frac{d}{dy} \left(\int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right)$$

First break up the doubly improper integral into two parts:

$$f_Y(y) \; = \; \frac{d}{dy} \left(\int_{-\sqrt{y}}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \; + \; \int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right)$$

Applying the fundamental theorem of calculus we get:

$$f_Y(y) \; = \; -\frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{y})^2}{2}} \cdot \frac{-1}{2\sqrt{y}} \; + \; \frac{1}{\sqrt{2\pi}} e^{-\frac{\sqrt{y}^2}{2}} \cdot \frac{1}{2\sqrt{y}}$$

Collecting the exponential terms gives:

$$f_Y(y) \; = \; \frac{1}{\sqrt{2\pi}\sqrt{y}} e^{\frac{-y}{2}}$$

The random variable Y has the *chi-square* distribution (with 1 degree of freedom). You can verify with maple that

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\sqrt{y}} e^{-\frac{y}{2}} \; = \; 1$$

Example 2 Here is a more extreme example. Suppose as before $X \sim N(0,1)$, but this time we are interested in the random variable $Y = \cos X$. In this case the transform $g(x) = \cos x$ maps $\mathbb{R} \to [-1,1]$ and the inverse image of $[-1,y]$ consists of an infinite number of intervals, centered at the points x_n on the x-axis with $\cos x = -1$,

$$x_n \; = \; (2n-1)\pi, \quad n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Each interval has width $2(\pi - \cos^{-1} y)$, so the inverse image under f of $[-1,y]$ is the union of the intervals,

$$g^{-1}[-1,y] \; = \; \bigcup_{n \in \mathbb{Z}} [(2n-2)\pi + \cos^{-1} y, 2n\pi - \cos^{-1} y]$$

Example 3 Suppose $X \sim N(0,1)$ as before, but this time we are interested in the random variable

$$Y \; = \; e^X$$

The transform function $g(x) = e^x$ is one-to-one, and its inverse is $g^{-1}(y) = \ln y$. The density function approach can be used, so the pdf of Y is

$$h(y) \;=\; f\left[g^{-1}(y)\right]\left|\frac{d}{dy}g^{-1}(y)\right| \;=\; f(\ln y)\frac{1}{y}$$

$$h(y) = \frac{1}{\sqrt{2\pi}y}e^{-\frac{(\ln y)^2}{2}}$$

In this case Y is said to have a *lognormal* distribution.

We can use the CDF approach with this transform. Note that $g : \mathbb{R} \to (0, \infty)$. Because the transform is one-to-one, it has an inverse, and consequently the inverse image of an interval is another interval, which might get stretched or compressed but will remain in one piece. We can discover what this interval is by evaluating g^{-1} at each of the endpoints of the Y interval, 0 and y.

For the CDF of Y, $F_Y(y)$, we are interested in intervals of the form $(0, y]$. We can find the inverse image under the invertible function g of this interval by finding its image under g^{-1}. Since $g^{-1}(y) = \ln y$), we can just take the natural logs of the endpoints (treating the lower endpoint as a limit of finite positive numbers). The inverse image of $(0, y)$ is $(-\infty, \ln y)$, and so

$$F_Y(y) \;=\; \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

again applying the fundamental theorem of calculus,

$$f_Y(y) \;=\; \frac{d}{dy}F_Y(y) \;=\; \frac{1}{\sqrt{2\pi}}e^{-\frac{(\ln y)^2}{2}} \cdot \frac{1}{y}$$

$$f_Y(y) \;=\; \frac{1}{\sqrt{2\pi}y}e^{-\frac{(\ln y)^2}{2}}$$

which is the same result we obtained with the density function approach.