MATRIX FORMULAS FOR COVARIANCE

1. Variance of Linear Combinations

Suppose $\vec{X} = (X_1, X_2, \ldots, X_n)$ is a vector of random variables.

We defined the **variance-covariance** matrix V of $\vec{X}$ to be the $n \times n$ symmetric matrix with:

$$\begin{aligned} v_{ii} &= \sigma_i^2 = \mathrm{Var}(X_i) \\ v_{ij} = v_{ji} &= \sigma_{ij} = \mathrm{Cov}(X_i, X_j), \quad i \neq j \end{aligned}$$

Let $\vec{r} = (r_1, r_2, \ldots, r_n)$ be a vector of constants.

The variance of the linear combination

$$\vec{r}'\vec{X} = r_1 X_1 + r_2 X_2 + \cdots + r_n X_n$$

is given by the **quadratic form**

$$\vec{r}' V \vec{r}$$

1.1. Examples.

Suppose $\vec{X}$ is a vector of two random variables

$$\vec{X} = (X_1, X_2)$$

with variance-covariance matrix

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

where $\sigma_1^2 = \mathrm{Var}(X_1)$, $\sigma_2^2 = \mathrm{Var}(X_2)$, and $\sigma_{12} = \mathrm{Cov}(X_1, X_2)$.

1.1.1. *Independent Random Variables.*

If X_1 and X_2 are independent, then $\mathrm{Cov}(X_1, X_2) = 0$, so the variance-covariance matrix of $\vec{X}$ is:

$$V = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

For the linear combination $X_1 + X_2$, $\vec{r} = (1, 1)$, and the variance of

$$\vec{r}'\vec{X} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_1 + X_2$$

is

$$\vec{r}'V\vec{r} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sigma_1^2 + \sigma_2^2$$

For the linear combination $X_1 + X_2$, $\vec{r} = (a, b)$, and the variance of

$$\vec{r}'\vec{X} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = aX_1 + bX_2$$

is

$$\vec{r}'V\vec{r} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\sigma_1^2 & b\sigma_2^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^2\sigma_1^2 + b^2\sigma_2^2$$

1.1.2. *Non-Independent Random Variables.* If X_1 and X_2 are **not** independent, the variance-covariance matrix of $\vec{X}$ is:

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

For the linear combination $X_1 + X_2$, $\vec{r} = (1, 1)$, and the variance of

$$\vec{r}'\vec{X} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_1 + X_2$$

is

$$\vec{r}'V\vec{r} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 + \sigma_{12} & \sigma_{12} + \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$$

For the linear combination $X_1 + X_2$, $\vec{r} = (a, b)$, and the variance of

$$\vec{r}'\vec{X} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = aX_1 + bX_2$$

is

$$\vec{r}'V\vec{r} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} a\sigma_1^2 + b\sigma_{12} & a\sigma_{12} + b\sigma_2^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\sigma_{12}$$

1.2. **Multivariate Linear Combinations.** It is possible to have a matrix of coefficients R multiplying $\vec{X}$,

$$R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The resulting linear combination is a vector of random variables,

$$\vec{Y} = R'\vec{X} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} aX_1 + cX_2 \\ bX_1 + dX_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

and the variance-covariance matrix of $R'\vec{X}$ is

$$R'VR$$

1.2.1. *Independent Random Variables.* If X_1 and X_2 are independent, they have zero covariance and the variance-covariance matrix of $\vec{X}$ is

$$V = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

Suppose

$$R = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

so that

$$\vec{Y} = R'\vec{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

The variance-covariance matrix of $\vec{Y}$ is

$$R'VR = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_2^2 \\ \sigma_1^2 & -\sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & \sigma_1^2 - \sigma_2^2 \\ \sigma_1^2 - \sigma_2^2 & \sigma_1^2 + \sigma_2^2 \end{bmatrix}$$

so

$$\begin{array}{lclcl} \mathrm{Var}(Y_1) & = & \mathrm{Var}(X_1 + X_2) & = & \sigma_1^2 + \sigma_2^2 \\ \mathrm{Var}(Y_2) & = & \mathrm{Var}(X_1 - X_2) & = & \sigma_1^2 + \sigma_2^2 \\ \mathrm{Cov}(Y_1, Y_2) & = & \mathrm{Cov}(X_1 + X_2, X_1 - X_2) & = & \sigma_1^2 - \sigma_2^2 \end{array}$$

1.2.2. *Non-Independent Random Variables.* The following property appears in the Averbach and Meta text on page III-4:

for *r.v.'s* X_1, X_2, X_3, and X_4,

$$\mathrm{Cov}(a_1X_1 + a_2X_2, b_1X_3 + b_2X_4) =$$

$$a_1b_1\mathrm{Cov}(X_1, X_3) + a_1b_2\mathrm{Cov}(X_1, X_4) + a_2b_1\mathrm{Cov}(X_2, X_3) + a_2b_2\mathrm{Cov}(X_2, X_4)$$

This is a special case of the above matrix formula, with

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \quad R = \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ 0 & b_1 \\ 0 & b_2 \end{bmatrix} \quad R'\vec{X} = \begin{bmatrix} a_1X_1 + a_2X_2 \\ b_1X_3 + b_2X_4 \end{bmatrix}$$

Then

$$R'VR = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \sigma_{24} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 & \sigma_{34} \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_4^2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ 0 & b_1 \\ 0 & b_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1\sigma_1^2 + a_2\sigma_{12} & a_1\sigma_{12} + a_2\sigma_2^2 & a_1\sigma_{13} + a_2\sigma_{23} & a_1\sigma_{14} + a_2\sigma_{24} \\ b_1\sigma_{13} + b_2\sigma_{14} & b_1\sigma_{23} + b_2\sigma_{24} & b_1\sigma_3^2 + b_2\sigma_{34} & b_1\sigma_{34} + b_2\sigma_4^2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ 0 & b_1 \\ 0 & b_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1^2\sigma_1^2 + 2a_1a_2\sigma_{12} + a_2^2\sigma_2^2 & a_1b_1\sigma_{13} + a_1b_2\sigma_{14} + a_2b_1\sigma_{23} + a_2b_2\sigma_{24} \\ a_1b_1\sigma_{13} + a_1b_2\sigma_{14} + a_2b_1\sigma_{23} + a_2b_2\sigma_{24} & b_1^2\sigma_3^2 + 2b_1b_2\sigma_{34} + b_2^2\sigma_4^2 \end{bmatrix}$$

The off-diagonal entries in $R'VR$ match the formula from property 4 of Averbach and Mehta. However, this formula only works for the specific R matrix in this example.

The matrix formula $R'VR$ works for any R and V.