AVERBACH AND MEHTA 3.6 EXERCISES - #8

8) If $\text{Var}(X_1) = \text{Var}(X_2) = 1$ and $\rho(X_1, X_2) = 2/3$, find the value of k for which

$$\rho(X_1, X_1 + kX_2) = \frac{2}{3}$$

a) -3/4 b) -2/3 c) 2/3 d) 3/4 e) 1

Solution: Since the variances of X_1 and X_2 are both 1, $\text{Cov}(X_1, X_2) = \rho(X_1, X_2)$, and the variance-covariance matrix of $[X_1, X_2]$ is

$$V = \begin{bmatrix} 1 & 2/3 \\ 2/3 & 1 \end{bmatrix}$$

and the transform matrix is

$$A = \begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix}$$

and the variance-covariance matrix of $(X_1, X_1 + kX_2)$ is

$$\begin{aligned} A'VA &= \begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} \begin{bmatrix} 1 & 2/3 \\ 2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{2}{3} \\ 1 + \frac{2}{3}k & \frac{2}{3} + k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 + \frac{2}{3}k \\ 1 + \frac{2}{3}k & 1 + \frac{4}{3}k + k^2 \end{bmatrix} \end{aligned}$$

So $\rho(X_1, kX_1 + X_2)$ is

$$\rho(X_1, kX_1 + X_2) = \frac{2}{3} = \frac{1 + \frac{2}{3}}{\sqrt{1}\sqrt{1 + \frac{4}{3} + k^2}}$$

so

$$\left(1 + \frac{2}{3}k\right)^2 = \frac{4}{9}\left(1 + \frac{4}{3}k + k^2\right)$$

and therefore

$$1 + \frac{4}{3}k + \frac{4}{9}k^2 = \frac{4}{9} + \frac{16}{27}k + \frac{4}{9}k^2$$

and

$$\frac{15}{27} + \frac{36}{27}k - \frac{16}{27}k \ = \ 0$$

so

$$15 + 20k = 0 \quad \text{and} \quad k \ = \ -\frac{15}{20} \ = \ -\frac{3}{4}$$