3.6 Exercises Problem 4

Gene Quinn

3.6 Problem 4

Which of the following is the mgf of a random variable with pdf

$$\frac{(e-1)}{e^x}, \quad x = 1, 2, 3, \ldots$$

a) $\frac{1}{1-t}$ b) $\frac{(e-1)e^t}{e^{t+1}-1}$ c) $\frac{e-1}{e^{t+1}-1}$ d) $\frac{(e-1)e^{t-1}}{1-e^{t-1}}$ e) $\frac{e^t}{e-e^t}$

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By definition the mgf is

$$M_x(t) = \mathsf{E}(e^{tx}) = \sum_{x=1}^{\infty} e^{tx}\left(\frac{e-1}{e^x}\right) = (e-1)\sum_{x=1}^{\infty} e^{(t-1)x}$$

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$$= (e-1) \left[\left(e^{t-1}\right) + \left(e^{t-1}\right)^2 + \left(e^{t-1}\right)^3 + \cdots \right]$$

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$$= (e-1)\left[\left(e^{t-1}\right) + \left(e^{t-1}\right)^2 + \left(e^{t-1}\right)^3 + \cdots\right]$$

$$= (e-1)(e^{t-1})\left[1 + \left(e^{t-1}\right) + \left(e^{t-1}\right)^2 + \left(e^{t-1}\right)^3 + \cdots\right]$$

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$$=\; (e-1)\left[\left(e^{t-1}\right) + \left(e^{t-1}\right)^2 + \left(e^{t-1}\right)^3 + \cdots\right]$$

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Note that the series inside the square brackets is a geometric series, and for values of t near zero, the ratio of successive terms e^{t-1} is less than 1.

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As a result, the infinite series converges and its sum is

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In this case, $r = e^{t-1}$ and so the sum is

$$\sum_{i=0}^{\infty} \left(e^{t-1}\right)^i = \frac{1}{1-e^{t-1}}$$

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So we can rewrite the expression

$$= \ (e-1)(e^{t-1})\left[1+\left(e^{t-1}\right)+\left(e^{t-1}\right)^2+\left(e^{t-1}\right)^3+\cdots\right]$$

as

$$M_x(t) \ = \ (e-1)(e^{t-1})\left(\frac{1}{1-e^{t-1}}\right)$$

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as

$$M_x(t) = (e-1)(e^{t-1})\left(\frac{1}{1-e^{t-1}}\right)$$

The final answer is

$$M_x(t) = \mathsf{E}(e^{tx}) = \frac{(e-1)\,e^{t-1}}{1-e^{t-1}}$$