AVERBACH AND MEHTA 3.6 EXERCISES - #34

34) If $\mathrm{E}(X) = 4$, $\mathrm{Var}(X) = 2$, $\mathrm{E}(Y) = 3$, $\mathrm{Var}(Y) = 18$, and $\rho(X, Y) = 1/2$, then $\mathrm{Cov}(X + 2, X + 2Y)$ is:

a) 9 b) 12 c) 8 d) 4 e) 6

Solution: The variance-covariance matrix of $[X, Y]$ is

$$\begin{bmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Var}(Y) \end{bmatrix}$$

From the given information,

$$\mathrm{Cov}(X, Y) = \rho(X, Y)\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)} = \frac{1}{2}\sqrt{2}\sqrt{18} = \frac{1}{2}\sqrt{36} = 3$$

so the variance-covariance matrix of $[X, Y]$ is

$$V = \begin{bmatrix} 2 & 3 \\ 3 & 18 \end{bmatrix}$$

The transformation matrix is

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

so that

$$A' \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ X + 2Y \end{bmatrix}$$

and the variance-covariance matrix of the transformed variables $[X, X + 2Y]$ is

$$\begin{aligned} A'VA &= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 18 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ 8 & 39 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 8 & 86 \end{bmatrix} \end{aligned}$$

Adding a constant to a random variable does not change its variance or covariance, so

$$\mathrm{Cov}(X + 2, X + 2Y) = \mathrm{Cov}(X, X + 2Y) = 8$$