AVERBACH AND MEHTA 3.6 EXERCISES - #23

23) Let X, Y, and Z be *r.v.'s* and let a, b and c be nonzero constants. What is $\text{Cov}(aX + bY + c, Z)$?

a) $a\text{Cov}(X, Z) + b\text{Cov}(Y, Z)$ b) $a\text{Cov}(X, Z) + b\text{Cov}(Y, Z) + c$
c) $a\text{Cov}(X, Z) + b\text{Cov}(Y, Z) + c\text{Var}(Z)$ d) $|a|\text{Cov}(X, Z) + |b|\text{Cov}(Y, Z)$

e) $|a|\text{Cov}(X, Z) + |b|\text{Cov}(Y, Z) + |c|$

Solution: The variance-covariance matrix of $[X, Y, Z]$ is

$$V = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix}$$

and the transform matrix is

$$A = \begin{bmatrix} a & 0 \\ b & 0 \\ 0 & 1 \end{bmatrix}$$

and the variance-covariance matrix of the transformed variables is

$$A'VA = \begin{bmatrix} a & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \\ 0 & 1 \end{bmatrix}$$

The result of this multiplication is the 2×2 variance-covariance matrix of the random vector $(aX + bY, Z)$.

We can save some time by just calculating the entry we need. The off-diagonal entries in this matrix are both $\text{Cov}(aX + bY, Z)$, so we'll just calculate enough of the matrix product to find this.

$$A'VA = \begin{bmatrix} a & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cdot & \cdot & \cdot \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cdot & \cdot \\ a\sigma_{xz} + b\sigma_{yz} + 0\sigma_z^2 & \cdot \end{bmatrix}$$

So the answer is $a\sigma_{xz} + b\sigma_{yz}$ or, in the notation used in the text,

$$\mathrm{Cov}(aX + bY, Z) = a\mathrm{Cov}(X, Z) + b\mathrm{Cov}(Y, Z)$$

Since adding a constant does not change the variance or covariance of a random variable, we can write the final answer as

$$\mathrm{Cov}(aX + bY + c, Z) = a\mathrm{Cov}(X, Z) + b\mathrm{Cov}(Y, Z)$$