AVERBACH AND MEHTA 3.6 EXERCISES - #13

13) Let X and Y be continuous *r.v.'s* with the same pdf and let $Z = kX - Y$. If $\rho(X,Z) = 1/3$, what is the value of k?

a) $1/\sqrt{2}$ b) $1/3$ c) $1/2$ d) $1/\sqrt{2}$ e) $1/\sqrt{3}$

Solution: The variance-covariance matrix of $[X,Y]$ is

$$\begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

and the transform matrix is

$$A = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$$

and the variance-covariance matrix is

$$\begin{aligned} A'VA &= \begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \sigma^2 & 0 \\ k\sigma^2 & -\sigma^2 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \sigma^2 & k\sigma^2 \\ k\sigma^2 & k^2\sigma^2 + \sigma^2 \end{bmatrix} \end{aligned}$$

$\rho(X,Z)$ is given by

$$\rho(X,Z) = \frac{1}{3} = \frac{k\sigma^2}{\sqrt{\sigma^2}\sqrt{(k^2+1)\sigma^2}} = \frac{k}{\sqrt{k^2+1}}$$

so

$$\frac{1}{3} = \frac{k}{\sqrt{k^2+1}}$$

and therefore

$$k^2 + 1 = 9k^2 \quad \text{so} \quad k = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$