RATIONAL FUNCTIONS

1. Rational Functions

1.1. **Definition.** A **rational function** is a function of the form

$$f : \mathbb{R} \to \mathbb{R} \quad \text{by} \quad f(x) \;=\; \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

In other words, a rational function is the ratio of two polynomials.

Rational functions are useful for representing phenomena that cannot be adequately described by a polynomial.

In particular, rational functions

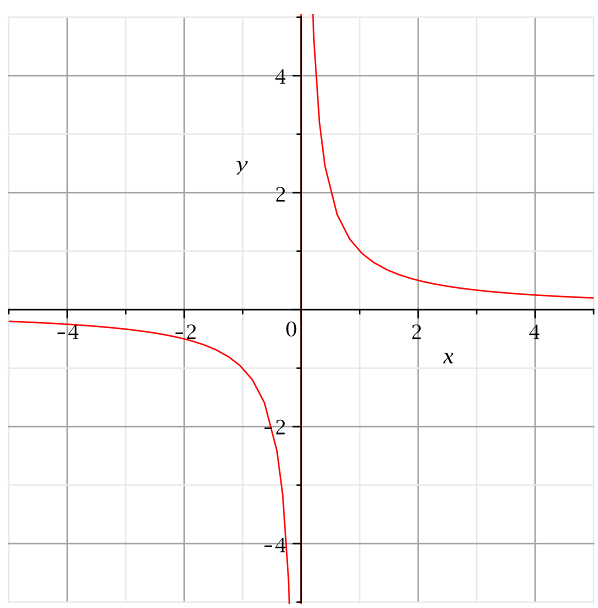
- can have vertical asymptotes
- can have horizontal asymptotes

Example 1.1. *The simplest rational function is the reciprocal function*

$$f(x) \;=\; \frac{1}{x}$$

Here we view the numerator as a polynomial of degree zero, and the denominator as a polynomial of degree one.

This function has a vertical asymptote at $x = 0$ *and a horizontal asymptote at* $y = 0$*:*

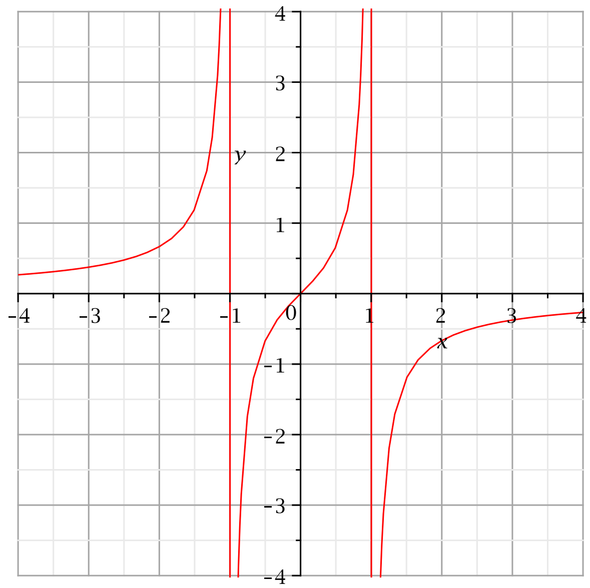


Example 1.2. *For the rational function*

$$f(x) = \frac{1}{1-x^2}$$

we view the numerator as a polynomial of degree one, and the denominator as a polynomial of degree two.

This function has a vertical asymptotes at $x = \pm 1$ and a horizontal asymptote at $y = 0$:



1.2. **Domain and Range.** Since a rational function is the quotient of two polynomials, the domains of the numerator and denominator are both $\mathbb{R}$.

As with the quotient of two arbitrary functions, we have to exclude values of x that make $Q(x) = 0$.

Since x will be assuming only real values, this means we have to exclude the **real** roots of Q:

$$f(x) = \frac{P(x)}{Q(x)} \quad D_f = \mathbb{R} \setminus \{x : Q(x) = 0\}$$

If the denominator Q has only complex roots, the domain of the rational function is $\mathbb{R}$.

It is difficult to make a general statement about the range of a rational function, because the behavior of these functions is so diverse. It is often useful to plug in values and evaluate the function, or use technology to graph the function to get a rough idea of the range.

Example 1.3. *The single root of* $Q(x) = x$ *is* $x = 0$, *so if*

$$f(x) = \frac{1}{x} \quad \textit{The domain of } f \textit{ is: } \mathbb{R} \setminus \{0\}$$

Example 1.4. *The roots of* $Q(x) = 1 - x^2$ *are* $x = \pm 1$, *so if*

$$f(x) = \frac{1}{1 - x^2} \quad \textit{The domain of } f \textit{ is: } \mathbb{R} \setminus \{-1, 1\}$$

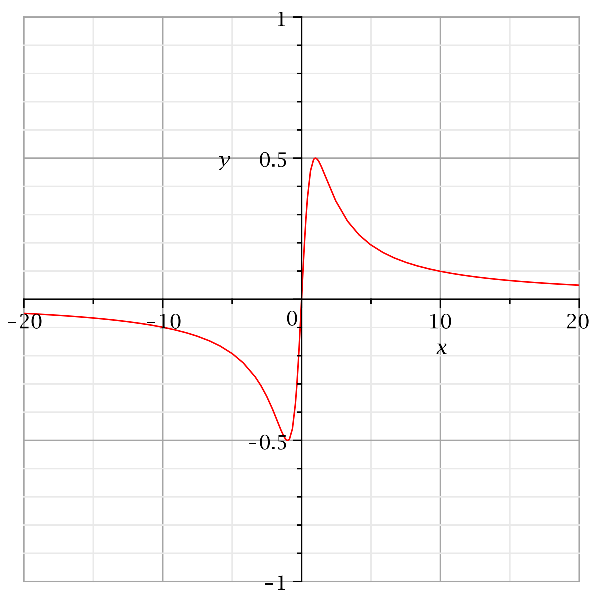
Example 1.5. *For the rational function*

$$f(x) = \frac{x}{x^2 + 1}$$

the denominator has no real roots, so if

$$f(x) = \frac{x}{x^2 + 1} \quad \textit{The domain of } f \textit{ is: } \mathbb{R}$$

The graph of this function has no vertical asymptotes, but the x*-axis is a horizontal asymptote.*

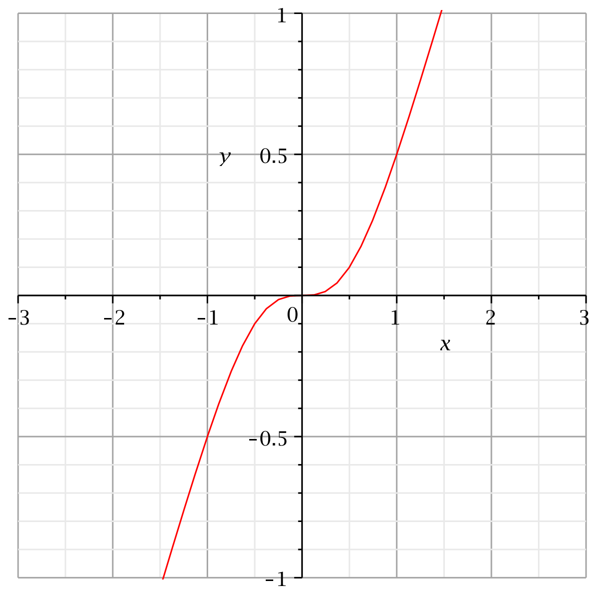


Example 1.6. *A rational function does not have to have vertical or horizontal asymptotes. The rational function*

$$f(x) = \frac{x^3}{x^2+1}$$

has a denominator with no real roots, so its domain is $\mathbb{R}$.

The graph of this function has no vertical asymptotes and no horizontal asymptotes.



1.3. **Asymptotic Behavior.** One of the characteristics of rational functions that makes them useful is the wide variety of behavior they can exhibit.

Consequently, it is difficult to make a general statement about their asymptotic behavior.

A rational function may go off to ∞ or $-\infty$ as x becomes large in either direction, or it may approach some finite value.

1.4. **Important Characteristics.** The characteristics of rational functions are rather diverse.

A rational function may:

- Behave much like a polynomial, with no horizontal or vertical asymptotes
- Behave differently than a polynomial by approaching $\pm\infty$ at some finite x
- Behave differently than a polynomial by approaching a finite value as x tends to $\pm\infty$