POWER FUNCTIONS

1. Power Functions

1.1. **Definition.** A **power function** is a function of the form

$$f : \mathbb{R} \to \mathbb{R} \quad \text{by} \quad y \;=\; f(x) \;=\; x^a$$

where a is a real number.

We will confine our discussion to the case where a is **rational**, because that allows us to express x^a in terms of integer powers and roots.

We will also consider only positive values, postponing consideration of negative powers until the section on rational functions.

If $a > 0$ is rational, it can be written as a quotient of two integers m and n,

$$a \;=\; \frac{m}{n} \quad m, n \in \mathbb{Z}, \quad n \neq 0$$

then, from our powers and roots section results, every power function with a positive rational exponent can be written as:

$$x^a \;=\; x^{\frac{m}{n}} \;=\; \sqrt[n]{x^m} \;=\; (\sqrt[n]{x})^m$$

1.2. **Domain and Range.** For any complex numbers z and u, the quantity

$$z^u$$

is perfectly well defined. The difficulties arise when we restrict ourselves to **real-valued** functions, for example, requiring

$$y \;=\; \sqrt[n]{x}$$

to be a real number. If $x \geq 0$, there are no problems, but if $x < 0$, there may not be any real solutions to this equation.

It is useful in this situation to think of the radicand as

$$\sqrt[n]{(-1) \cdot x}$$

for some $x > 0$. Then using the algebraic properties of powers and roots, we can write this as

$$((-1) \cdot x)^{\frac{1}{n}} \;=\; (-1)^{\frac{1}{n}} \, (x)^{\frac{1}{n}} \;=\; \sqrt[n]{-1} \cdot \sqrt[n]{x}$$

The problem now reduces to determining whether or not there is a real number u such that

$$u = \sqrt[n]{-1}$$

There are certainly complex numbers that make this equation true since any solution of the equation

$$u^n + 1 = 0$$

qualifies as an n^{th} root of -1.

As it turns out, if n is odd, we can always factor $u^n + 1$ into an expression of the form

$$u^n + 1 = (u+1)(u^{n-1} - u^{n-2} + u^{n-3} - \cdots + 1)$$

which shows that, if n is odd, -1 is always an n^{th} root of -1.

Example 1.1. *Find all solutions to the equation*

$$u^3 + 1 = 0$$

First factor out $(u+1)$, which is possible since n is odd:

$$u^3 + 1 = (u+1)(u^2 - u + 1)$$

Now apply the quadratic formula to the second factor,

$$u = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

so the solutions are

$$-1, \quad \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

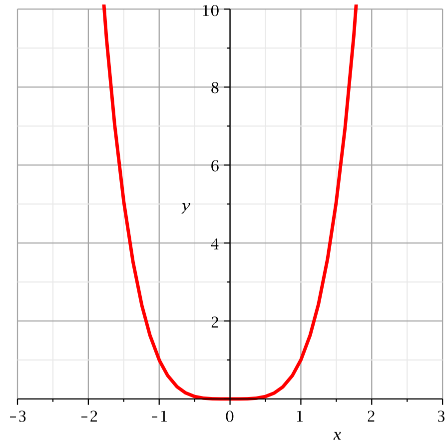
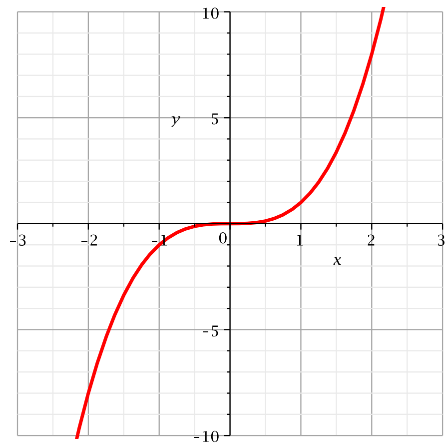
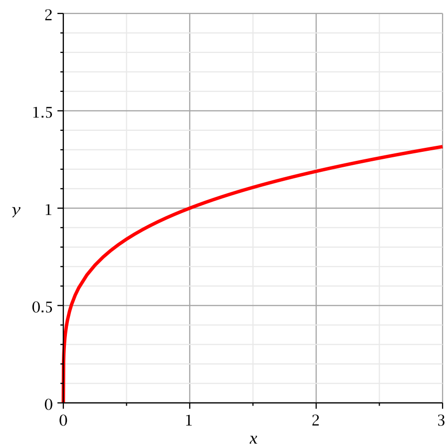
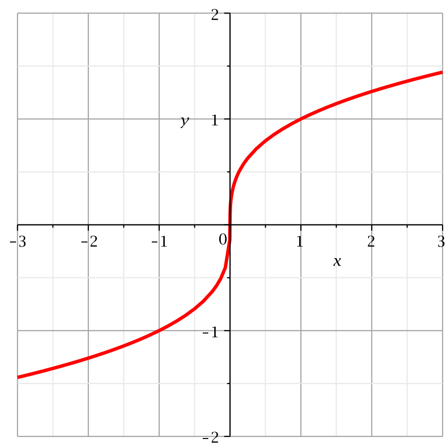
All of these numbers produce -1 when cubed, but only -1 is a real number. So if we are restricted to real solutions, we would write

$$\sqrt[3]{-1} = -1$$

As it turns out, the solutions of $u^n + 1 = 0$ depend on whether n is even or odd.

- If n is odd, -1 is the only real root. There are $n-1$ complex roots.
- If n is even, all of the roots are complex.

The properties of power functions are summarized in the following table for positive integers n:

x^n	n even	Domain: $\mathbb{R}$	Range: $[0, \infty)$	
x^n	n odd	Domain: $\mathbb{R}$	Range $\mathbb{R}$	
$\sqrt[n]{x}$	n even	Domain: $[0, \infty)$	Range: $[0, \infty)$	
$\sqrt[n]{x}$	n odd	Domain: $\mathbb{R}$	Range: $\mathbb{R}$	

1.3. **Asymptotic Behavior.** For any positive integer n,

- The values of x^n and $\sqrt[n]{x}$ tend to ∞ as x becomes large and positive.
- If n is odd, x^n and $\sqrt[n]{x}$ tend to $-\infty$ as x becomes large and negative
- If n is even, x^n tends to ∞ as x becomes large and negative
- x^n and $\sqrt[n]{x}$ have no vertical or horizontal asymotote

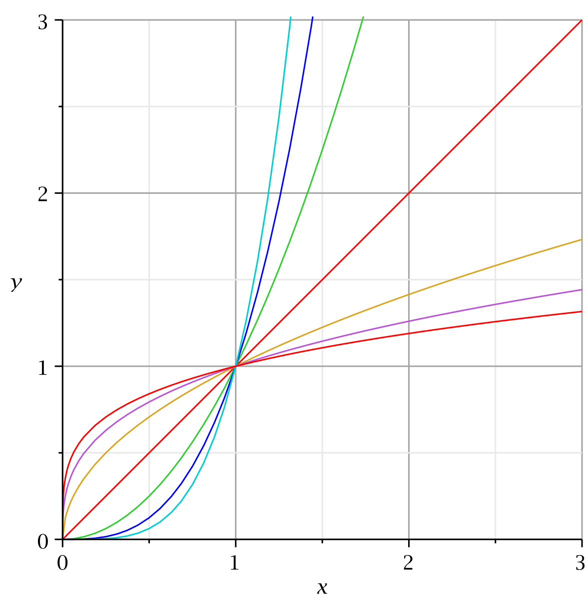
1.4. **Important Characteristics.** The graph of every power function passes through the origin $(0, 0)$ and the point $(1, 1)$.

If the domain includes negative real numbers and n is even, the graph passes through the point $(-1, 1)$.

If the domain includes negative real numbers and n is odd, the graph passes through the point $(-1, -1)$.

1.4.1. *Growth Rates.* The following figure contains the graphs of the following power functions in the first quadrant:

$$\sqrt[4]{x},\ \sqrt[3]{x},\ \sqrt{x},\ x,\ x^2,\ x^3,\ x^4$$



On the interval $(1, \infty)$, $\sqrt[4]{x}$ has the smallest value and x^4 has the largest, with the other power functions falling in between in order of increasing exponents.

$$\sqrt[4]{x} < \sqrt[3]{x} < \sqrt{x} < x < x^2 < x^3 < x^4$$

On the interval $(0,1)$, the order is reversed: the function with the smallest exponent has the largest value

$$x^4 < x^3 < x^2 < x < \sqrt{x} < \sqrt[3]{x} < \sqrt[4]{x}$$

This result is actually a bit more general. For any positive real numbers a, b, n, m with $m < n$, as x increases eventually

$$ax^m < bx^n$$

no matter how much larger a is than b.

Example 1.2. *As x becomes larger and larger, eventually*

$$10,000 \cdot x < \frac{x^2}{10,000}$$

In this case, we need to take $x > 10,000,000,000$