POLYNOMIALS

1. Polynomials

1.1. **Definition.** A **polynomial** is a function of the form

$$f : \mathbb{R} \to \mathbb{R} \quad \text{by} \quad P(x) \ = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are real constants and n is a nonnegative integer.

n is called the **degree** of the polynomial.

The constants $a_0, a_1, \ldots, a_n$ are called the **coefficients** of the polynomial.

A polynomial is a sum of power functions with nonnegative integer exponents.

- a polynomial of degree 0 $P(x) = a_0$ is a constant function
- a polynomial of degree 1 $P(x) = a_1 x + a_0$ is a linear function
- a polynomial of degree 2 $P(x) = a_2 x^2 + a_1 x + a_0$ is called a quadratic function
- a polynomial of degree 3 $P(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ is called a cubic function

Polynomials are among the most useful functions for mathematical modeling because they are relatively simple and can assume a great variety of shapes.

Example 1.1.

$$P(x) \ = \ x^3 - 2x^2 + x - 4$$

is a third degree or cubic polynomial.

Example 1.2.

$$P(x) \ = \ x^4$$

is a fourth degree polynomial. In an n^{th} degree polynomial, only a_n has to be nonzero.

1.2. **Domain and Range.** The domain of every polynomial is $\mathbb{R}$.

The range of every polonomial of **odd** degree is $\mathbb{R}$.

The range of a polynomial of **even** degree depends on the degree n and the sign of a_n:

- $\{a_0\}$ if $n = 0$ ($P(x)$ is a constant function)

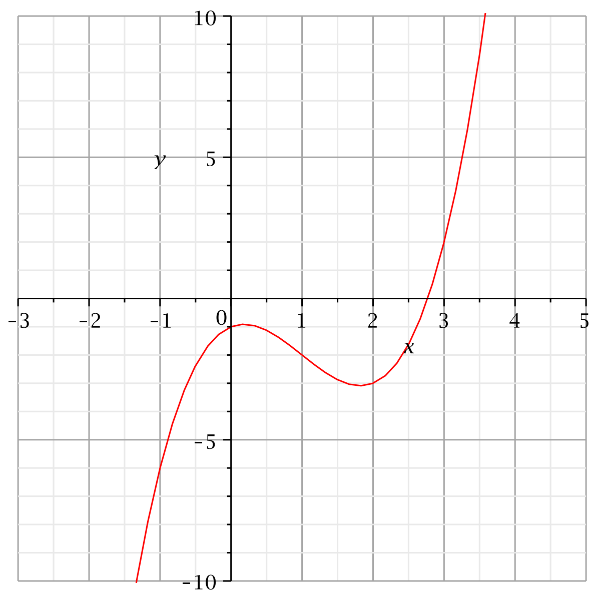
- $\{x : x \geq k\}$ if $n > 0$ and $a_n > 0$
- $\{x : x \leq k\}$ if $n > 0$ and $a_n < 0$

where k is a real constant.

Example 1.3. *The domain and range of the third degree polynomial*

$$P(x) \ = \ x^3 - 3x^2 + 2 - 1$$

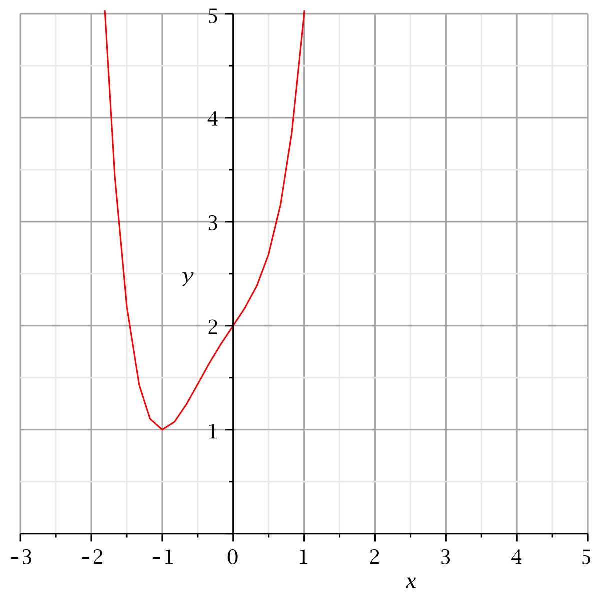
are both $\mathbb{R}$ *because* $n = 3$ *is odd.*



Example 1.4. *The domain of the fourth degree polynomial*

$$P(x) \ = \ x^4 + x^3 + x + 2$$

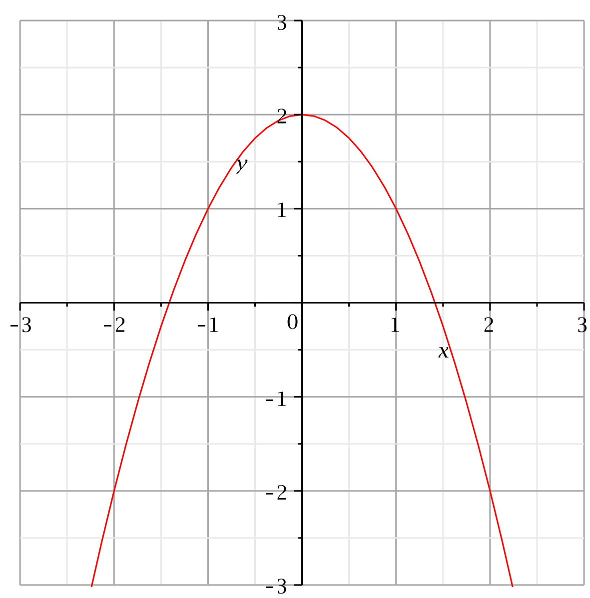
is $\mathbb{R}$. *The range has the form* $[k, \infty)$ *where* k *is a constant that depends on the coefficients. For this polynomilal, the range is* $[1, \infty)$.



Example 1.5. *The domain of the second degree polynomial*

$$P(x) \;=\; -x^2 + 2$$

is $\mathbb{R}$. *The range has the form* $(-\infty, k]$ *where* k *is a constant that depends on the coefficients. For this polynomilal, the range is* $(-\infty, 2]$.



1.3. **Asymptotic Behavior.** For any polynomial with $n > 0$,

- The value of $P(x)$ always tends to either ∞ or $-\infty$ as x becomes large and positive.
- The value of $P(x)$ always tends to either ∞ or $-\infty$ as x becomes large and negative
- The graph of $P(x)$ has no vertical or horizontal asymptotes

1.4. **Important Characteristics.** $P(x)$ is defined and finite for any real value of x.

The value of $P(x)$

- Does not tend to ∞ for any finite value of x
- Always tends to $\pm\infty$ as x approaches $\pm\infty$

Two polynomials

$$P_1(x) \;=\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$P_2(x) \;=\; b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

are equal for all values of x if and only if their coefficients are the same:

$$a_n = b_n, \quad a_{n-1} = b_{n-1}, \quad \ldots \quad a_1 = b_1, \quad a_0 = b_0$$

Of course, this also implies that they are of the same degree.

Solutions of the equation

$$P(x) \;=\; 0$$

are called the **roots** of the polynomial.

The following important result is known as the **fundamental theorem of algebra**:

If we allow complex solutions and count multiplicities, every polynomial of degree n has n roots.

Equivalently, we can say that every polynomial of degree n can be written as the product of n factors

$$k \cdot (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

and, possibly, a constant k where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of $P(x)$.

Example 1.6. *The roots of*

$$P(x) \ = \ x^2 - 2$$

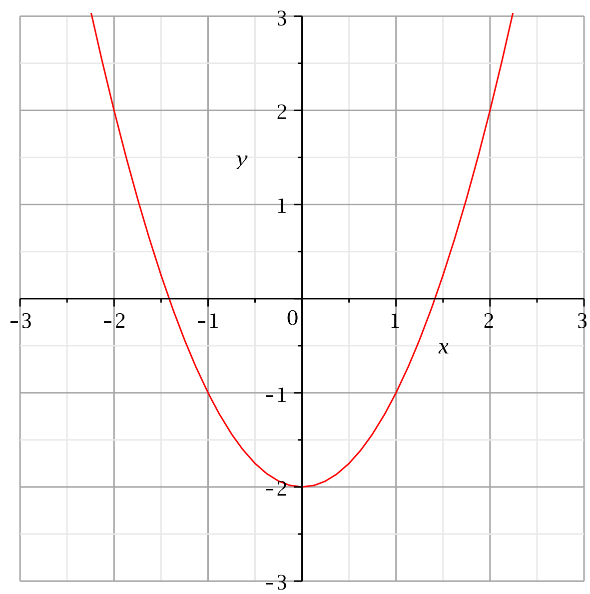
can be found using the quadratic formula:

$$x = \frac{0 \pm \sqrt{0 - 4 \cdot (-2)}}{2} \ = \ \pm\frac{\sqrt{8}}{2} \ = \ \pm\sqrt{2}$$

We can also think of factoring $P(x)$ into:

$$P(x) \ = \ x^2 - 2 \ = \ (x - \sqrt{2})(x + \sqrt{2})$$

so the roots are $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{x}$. The roots are the x-coordinates of the points where the graph crosses the x-axis.



Example 1.7. *The roots of*

$$P(x) \ = \ x^2 + 2x + 1$$

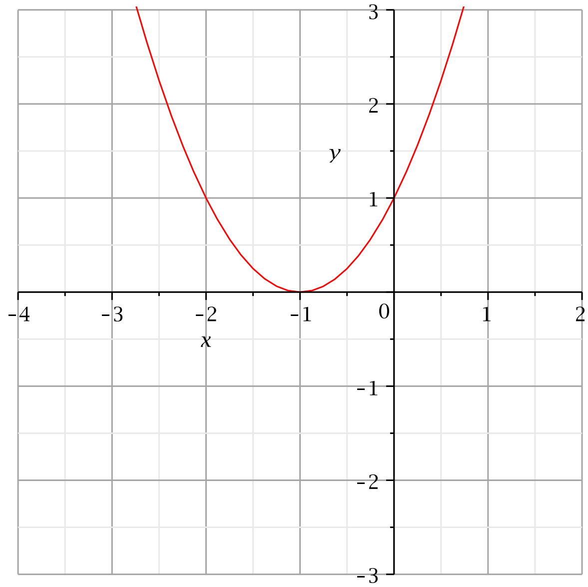
can be found using the quadratic formula:

$$x = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 1}}{2} \ = \ \frac{-2 \pm 0}{2} \ = \ -1$$

We can factor $P(x)$ into:

$$P(x) \ = \ x^2 + 2x + 1 \ = \ (x + 1)(x + 1)$$

$\lambda_1 = -1$ is a root with multiplicity 2 (the multiplicity is the number of identical factors the root has).



Example 1.8. *Find the roots of*

$$P(x) \;=\; x^3 - x^2 + 2x - 2$$

We do not have a formula analagous to the quadratic formula for cubics, but we can factor this polynomial into

$$P(x) \;=\; x^3 - x^2 + 2x - 2 \;=\; (x-1)(x^2+2)$$

This tells us that 1 is a root, and we can use the quadratic formula for the second factor,

$$x = \frac{0 \pm \sqrt{0 - 4 \cdot 2}}{2} \;=\; \frac{\pm\sqrt{-8}}{2} \;=\; \pm\frac{\sqrt{-8}}{\sqrt{4}} \;=\pm 2i$$

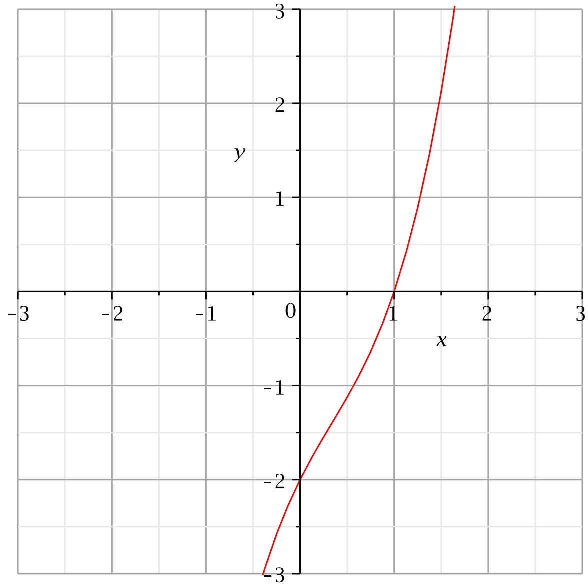
We can factor $P(x)$ fully into:

$$P(x) \;=\; x^3 - x^2 + 2x - 2 \;=\; (x-1)(x-2i)(x+2i)$$

so the roots are $\lambda_1 = 1$, $\lambda_2 = -2i$, and $\lambda_3 = 2i$.

This polynomial has one real root and two complex roots. If a polynomial has complex roots, they always occur in conjugate pairs.

The graph of this polynomial crosses the x-axis only once, at $x = 1$, because there is one real root.



Example 1.9. *Find a polynomial that has roots* -1, 1, *and* 2.

In this case, we write down the factors that produce these roots, and expand the product:

$$P(x) \;=\; (x+1)(x-1)(x-2) \;=\; (x^2-1)(x-2) \;=\; x^3-2x^2-x+2$$