LINEAR FUNCTIONS

1. Linear Functions

1.1. **Definition.** A **linear function** is a function of the form

$$f : \mathbb{R} \to \mathbb{R} \quad \text{by} \quad y \;=\; f(x) \;=\; mx + b$$

where m and b are real numbers.

1.2. **Domain and Range.** Regardless of the values of m and b, the domain of a linear function is always $\mathbb{R}$.

The range of a linear function is:

- $\mathbb{R}$ if $m \neq 0$
- $\{b\}$ if $m = 0$

The linear function with $m = 0$, defined by

$$f : \mathbb{R} \to \{b\} \quad \text{by} \quad y \;=\; f(x) \;=\; b$$

is sometimes called the **constant function** because it assigns the same value to every element of the domain.

Students are sometimes confused when asked to evaluate, say, $f(3)$ because there is nowhere to "plug in" 3 in the formula, but regardless of the value of x, the value of $f(x)$ is b.

In terms of our definition of a function as a domain, codomain, and a set of ordered pairs, for the constant function ($m = 0$)

- The domain is $\mathbb{R}$
- The codomain could be any set that includes $\{b\}$
- The set of ordered pairs is $\{(x, b) \;:\; x \in \mathbb{R}\}$

1.3. **Asymptotic Behavior.** With the exception of the special case $m = 0$, $f(x)$ always tends to either ∞ or $-\infty$ as the distance between x and the origin increases without bound.

If $m > 0$, $f(x)$ tends to ∞ as x becomes large in the positive direction, and to $-\infty$ for large negative values of x.

If $m < 0$, $f(x)$ tends to $-\infty$ for large positive values of x, and to ∞ for large negative values of x.

The vaule of $f(x)$ is defined and finite for any real number x, so the graph of a linear function has no vertical asymptotes.

Because $f(x)$ always tends to $\pm\infty$ for large values of x, there are no horizontal asymptotes either.

1.4. **Inverses.** Except for those with $m = 0$, every linear function has an inverse.

In general, the inverse is

$$f(x) \;=\; mx + b, \quad m \neq 0 \qquad f^{-1}(x) \;=\; \frac{x-b}{m}$$

1.5. **Important Characteristics.** The graph of a linear function is always a straight line.

- In the special case $m = 0$, the line is horizontal.
- In the special case $b = 0$, the line passes through the origin.
- If both m and b are zero, the line is the x-axis, with equation $y = 0$

A vertical line is not the graph of any linear function because it clearly fails the vertical line test.

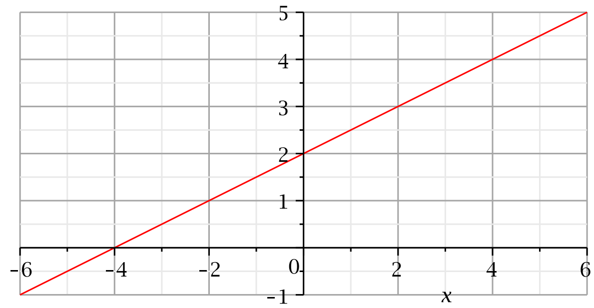
Every linear function is one-to-one, that is,

$$u \neq v \quad \text{implies} \quad f(u) \neq f(v)$$

If $m > 0$, f is an *increasing* function everywhere on its domain, which means that

$$x_1 > x_2 \quad \text{implies} \quad f(x_1) > f(x_2)$$

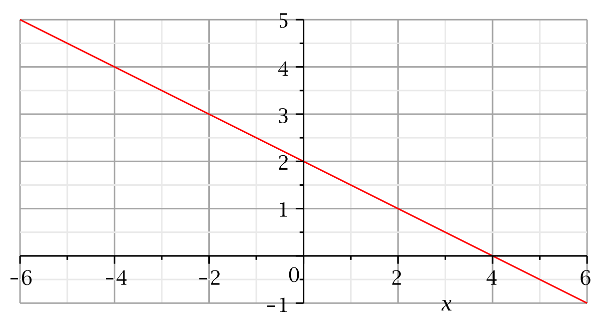
for any values $x_1, x_2 \in \mathbb{R}$.



If $m < 0$, f is a *decreasing* function everywhere on its domain, which means that

$$x_1 > x_2 \quad \text{implies} \quad f(x_1) < f(x_2)$$

for any values $x_1, x_2 \in \mathbb{R}$.



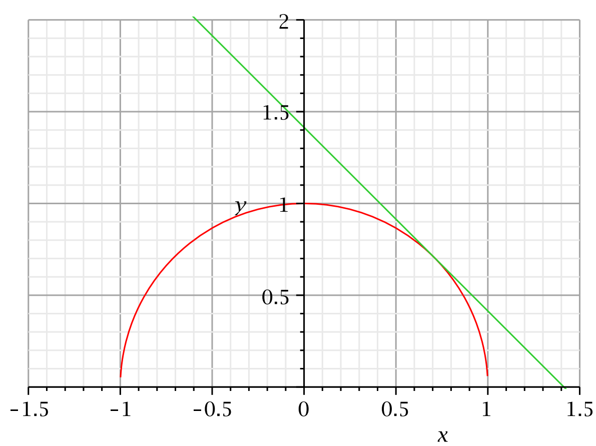
An important characteristic of a linear function is that its rate of change, expressed as the change in y per unit change in x, is constant and independent of the value of x.

Another way to say this is that the change in the y-coordinate per unit change in the x-coordinate is always the same and does not depend on what the actual x coordinate value is.

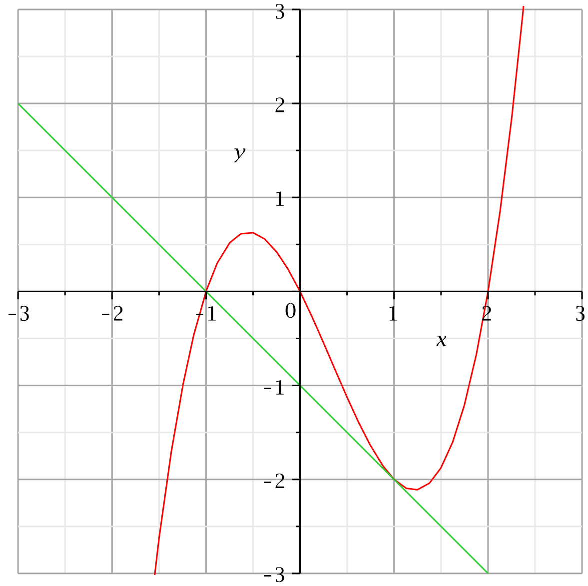
Linear functions are extremely useful for the following reasons:

- The relationship between x and y is inherently simple and easy to understand
- The values of m and b are relatively easy to determine empirically
- Over a short interval, most functions that describe physical phenomena can be approximated by a linear function

In geometry, we are introduced to the concept of a *tangent* line, usually a tangent to a circle, which is defined as a straight line that intersects a curve at a single point.



The central theme in the first semester of calculus is extending this idea to the broadest class of functions possible,



The linear function corresponding to the tangent line intersecting the graph of f at the point $(x, f(x))$ will play an important role in determining the instantaneous rate of change of the function, and in approximating the function in the viscinity of x.

1.6. **Linear Functions in Other Areas of Mathematics.** There is unfortunately an inconsistency in the way the term "linear function" is used in mathematics.

In calculus, a linear function is usually defined to have the form $f(x) = mx + b$ (i.e., any function whose graph is a straight line).

However, in abstract an linear algebra, a linear function is defined as a function having the two properties

$$f(u+v) \;=\; f(u)+f(v) \quad \text{and} \quad f(k\cdot x) \;=\; k\cdot f(x)$$

A function of the form $f(x) = mx + b$ satisfies the second definition only when $b = 0$. That is, only linear functions whose graphs pass through the origin satisfy the definition of a linear function used in many other areas of mathematics.