DEFINING A FUNCTION

1. Functions

Many consider the idea of a *function* to be the most important single concept in mathematics.

Informally, a function is a pairing that associates every element of a particular set, called the **domain**, with an element of another set, which some authors call the **codomain**.

An alternative terminology for a function is a *map* or *mapping* of the elements from the domain to the codomain. This terminology also reflects the fact that the central idea is an association between two sets.

A more precise definition of a function is the following:

Given two sets A and B, a *function* from A into B, denoted by

$$f \; : \; A \; \rightarrow B$$

is a subset of the set of all possible ordered pairs of the form

$$(x, y) \quad \text{where} \quad x \in A \;\; \text{and} \;\; y \in B$$

with the property that every element of A appears as the first entry of **exactly one** ordered pair.

The set A is called the *domain* of the function. The set B is sometimes called the *codomain*.

The set consisting of every possible ordered pair (x, y) with $x \in A$ and $y \in B$ is known as the **Cartesian product** of A and B (another reference to Rene Descartes) and is denoted by

$$A \times B \quad = \quad \{(x, y) \; : \; x \in A \text{ and } y \in B\}$$

Using this terminology, a function $f : A \rightarrow B$ from A into B is a subset of the Cartesian product of the domain A and codomain B with the property that every element of the domain appears as the first entry in exactly one ordered pair in the subset.

Example 1.1. *Suppose*

$$A \; = \; \{1, 2, 3\} \quad \textit{and} \quad B = \{3, 5, 7, 9\}$$

Then the Cartesian product $A \times B$ consists of 12 ordered pairs,

$$A \times B \quad = \quad \left\{ \begin{array}{cccc} (1,3) & (1,5) & (1,7) & (1,9) \\ (2,3) & (2,5) & (2,7) & (2,9) \\ (3,3) & (3,5) & (3,7) & (3,9) \end{array} \right\}$$

Any subset of these 16 ordered pairs that contains exactly one from each row is a function from A into B.

These include:

$$\begin{array}{ll} f_1 : A \to B & \{(1,3),(2,5),(3,7)\} \\ f_2 : A \to B & \{(1,9),(2,5),(3,7)\} \\ f_3 : A \to B & \{(1,5),(2,5),(3,3)\} \\ f_4 : A \to B & \{(1,7),(2,3),(3,3)\} \end{array}$$

The interpretation of this table is that $f_1(1) = 3,\ f_1(2) = 5,\ f_1(3) = 7$, with similar interpretations for the other functions.

The set of elements that appear as the second entry of an ordered pair is called the **range** of the function.

The range of f_1 is $\{3,5,7\}$, while the range of f_4 is $\{3,7\}$.

There is no requirement that every element of B appear in the selected ordered pairs, so

$$f_5 \ = \ \{(1,3),(2,3),(3,3)\}$$

is a perfectly valid function from A into B. It makes no difference that $f(1) = f(2) = f(3) = 3$, it just means that the range of this function consists of a single element of B.

With four choices for the ordered pair that starts with 1, four for the one that starts with 2, and four for the one that starts with 3, there are

$$4 \cdot 4 \cdot 4 \quad = \quad 64$$

possible functions from A into B.

Not every subset of $A \times B$ represents a function. The following sets of ordered pairs **do not** represent functions from A into B:

Subset	Reason
$\{(1,3),(2,5)\}$	Every element of the domain A must appear as the first element of an ordered pair (3 does not)
$\{(1,3),(1,5),(2,7),(3,9)\}$	No element of A can appear in more than one ordered pair (1 appears twice)

In summary, there are three components to the definition of a function $f : A \to B$,

- A set A called the**domain**
- A set B called the **codomain**
- A subset of $A \times B$ with each element of A appearing in exactly one ordered pair

1.1. **Ways of Representing Functions.** We will consider several ways of representing functions.

In all cases, a specification of $f : A \to B$ must state what the domain A and codomain B are.

However, when it comes to declaring what the subset of $A \times B$ associated with f is, practical considerations dictate the need for several options other than listing the elements of the subset as we did earlier.

1.1.1. *Representing a Function as a Table of Values.* In retrospect, the first way to represent a function is to list the ordered pairs in the subset of $A \times B$ that appears in the function definition.

Of course, this is only practical when the domain A is a relatively small finite set, as our example was.

Less formally, the function may simply be represented as a table. For our sample function f_1, the table is:

x	f(x)
1	3
2	5
3	7

We still have to specify the domain and codomain, although the first column of the table will always contain the entire domain.

This method is often used when the domain and codomain represent pairs of physical measurements.

1.1.2. *Representing a Function Verbally.* Sometimes the easiest to describe a function is in words.

Example 1.2. *Let f be a function from the integers $\mathbb{Z}$ into the natural numbers $\mathbb{N}$ defined by the following:*

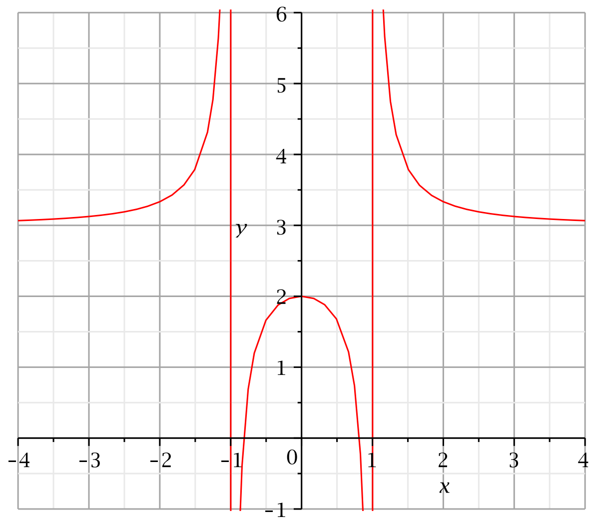
For any $z \in \mathbb{Z}$, $f(z)$ is the smallest even natural number that is larger than z.

Note that by this rule, the function value for any integer less than 2 is 2. After that, the next few integers follow the pattern:

x	$f(x)$
2	4
3	4
4	6
5	6
6	8
⋮	⋮

Because the domain is not finite, it is not practical to list all of the $(x, f(x))$ pairs in the function.

1.1.3. *Representing a Function Graphically.* Sometimes a graph is a suitable representation of a function. As long as the resolution permits determining the value of the function with sufficient accuracy, this method can be practical. For example, we might consider the function defined by the graph:



When a function is specified by a graph, the domain and codomain are determined by the range of values on the graph.

1.1.4. *Representing a Function by a Formula.* Perhaps the most familiar method of specifying a function is to use a formula like

$$f(x) \;=\; x^2$$

In some cases, the domain and codomain will be explicitly given.

Example 1.3. *The statement*

$$f : [0,1) \to \mathbb{R} \;\; by \;\; y = x^2$$

defines a function whose domain is the interval $[0,1)$ *and whose codomain is the real numbers. The ordered pairs specifying the association of elements in the domain and codomain could be written as:*

$$\{(x,y) \;:\; x \in [0,1) \;\; and \;\; y = x^2\}$$

Most of the time, when a formula is used to represent a function, the domain and codomain are not explicitly stated.

Normally, we can just take the codomain to be the real numbers $\mathbb{R}$ or, equivalently, the interval $(-\infty, \infty)$.

The domain is trickier. The convention for determining the domain when the function is represented by a formula is the following:

Unless otherwise stated, the domain is considered to be the set of all real numbers for which the formula produces a real number.