DOMAIN, CODOMAIN, AND RANGE

1. Domain, Codomain, and Range

We stated earlier that there are three components to the definition of a function $f : A \to B$,

- A set A called the **domain**
- A set B called the **codomain**
- A subset of $A \times B$ with each element of A appearing in exactly one ordered pair

It is important to realize that the domain and codomain are part of the definition of a function.

1.1. The Domain.

Example 1.1. *The function definitions*

$$f : [0, \infty) \to \mathbb{R} \quad by \quad f(x) = x^2$$

and

$$f : \mathbb{R} \to \mathbb{R} \quad by \quad f(x) = x^2$$

both use the same formula $f(x) = x^2$, but they have different domains so they represent ***different functions****.*

The way the domain and codomain are specified depends on how the function definition is conveyed. The following table lists the usual way of specifying the domain and codomain:

How function is defined	How domain and codomain are specified
By a table of values	The column listing the independent variable
By a verbal description	As part of the description
By a graph	By the scale on the graph
By a formula	The domain is the set of inputs that produce a real number

In the last case, a function defined by a formula, the domain and codomain are usually not specified.

We can usually assume that the codomain is the set of real numbers $\mathbb{R}$, but we have to use the formula and our knowledge of common functions to determine the domain.

Example 1.2. *The domain of a function defined by a formula is the set of values that produce a real number when plugged into the formula.*

If a function is defined by the formula

$$f(x) \;=\; x^2$$

then the domain is the set of all real numbers x for which x^2 is a real number.

By the closure property, $x^2 = x \cdot x$ is a real number no matter what x is, so the domain of this function is $\mathbb{R}$.

In this course we will encounter two situations in which a formula does not produce a real number.

The first, and most common, is a quotient whose denominator is zero.

Example 1.3. *If a function is defined by the formula*

$$f(x) \;=\; \frac{1}{x}$$

then the domain is the set of all real numbers x for which $\frac{1}{x}$ is a real number.

From the field properties, recall that every real number x except zero has a multiplicative inverse $1/x$, so the domain of this function is all real numbers except zero. The notation for this is

$$\mathbb{R} \setminus 0$$

which stands for the set of real numbers excluding zero, read "$\mathbb{R}$ setminus 0".

The second situation is a root of even order with a negative quantity under the radical, like $\sqrt{-1}$.

Example 1.4. *If a function is defined by the formula*

$$f(x) \;=\; \sqrt{x}$$

then the domain is the set of all real numbers x for which $\sqrt{x}$ is a real number.

We know that sqrtx is a real number only when $x \geq 0$, so the domain of the function with formula $f(x) = \sqrt{x}$ is

$$\{x \;:\; x \geq 0\} \quad \textit{or in interval notation} \quad [0, \infty)$$

For a few common functions such as the natural log function ln, the domain is specified in the standard definition of the function. $\ln(x)$ is defined only for positive values of x, so its domain is $(0, \infty)$.

1.2. **The Codomain.** As noted previously, we can usually safely assume that the codomain is the set of all real numbers $\mathbb{R}$.

At the very least, the codomain has to include the function value of every element in the domain. Equivalently, the range must be a subset of the codomain.

1.3. **The Range.** The **range** of a function is the set of values that $f(x)$ assumes for some x in the domain of f.

If we think of the function as consisting of a set of ordered pairs, the range is the set of all elements that appear as the second entry of one or more ordered pairs.

Usually the range can be determined by some combination of computation, examining the graph, and considering the algebraic properties of the formula.

Example 1.5. *Find the range of the function defined by the formula*

$$f(x) \;=\; x^2 - 2$$

Often it is useful to try to find the largest and smallest values of $f(x)$. Since the smallest value possible for x^2 is zero, the smallest value for $f(x)$ is $0-2$ or -2. As x becomes large in either the positive or negative direction, x^2 becomes large without bound.

Based on these considerations, the range of f is $[-2, \infty)$.

Example 1.6. *Find the range of the function defined by the formula*

$$f(x) \;=\; \frac{1}{\sqrt{4+x}}$$

The minimum of the square root function $\sqrt{z}$ occurs when $z = 0$, so the minimum of $\sqrt{4+x}$ occurs when $4 + x = 0$ or, equivalently, $x = -4$, in which case the value is zero.

Since we are dividing by $\sqrt{4+x}$, we cannot allow zero, but $\sqrt{4+x}$ is considered positive, and as it approaches zero, its reciprocal becomes large without bound.

The larger x becomes, the smaller its reciprocal becomes. However, it is always positive, so it can approach, but not equal zero.

This means the range is $(0, \infty)$.

The range of the special function $\ln(x)$ is $(-\infty, \infty)$. As $x \to 0$, $\ln(x)$ approaches $-\infty$. As $x \to \infty$, $\ln(x)$ becomes large without bound (i.e., approaches ∞).