1. Project 5

Many biological populations (for example, salmon) exhibit what is known as a **recruitment cycle** in which the population at each stage is a function of the population size at the previous stage,

$$x_{n+1} = f(x_n)$$

but the form of the function f is determined by a continuous birth and death process, usually constructed from assumptions on the births and deaths occurring continuously in the intervals between stages.

Models of this type are called *metered models*

In many species of fish, there is an annual birth process followed by a continuous death rate until the next birth cycle. At the begining of the next cycle, the survivors from the previous cycle make up the population. Populations of this type are natural candidates for metered models.

Assuming constant fertitility α, if B_n is the number of births in the n^{th} cycle, which has x_n adults at the start, the number of births is

$$B_n = \alpha x_n$$

Between birth times, we assume a per capita death rate $\phi(z)$ that is a function of the number of survivors from the B_n newborn members (which implicitly assumes that adults do not survive beyond the current cycle). Then if $z(t)$ is the number surviving at time t, we assume the per capita death rate $\phi(z)$ satisfies the differential equation:

$$\frac{dz}{dt} = -z\phi(z)$$

The *recruitment* R_n is the value for $t = T$ of the solution to the initial value problem:

$$\frac{dz}{dt} = -z\phi(z), \quad z(0) = B_n = \alpha x_n$$

which satisfies the equation

$$\int_{\alpha x_n}^{R_n} \frac{dz}{z\phi(z)} = -T$$

and implies that the function f in the metered model

$$x_{n+1} = f(x_n)$$

satisfies the equation

$$\int_{\alpha x_n}^{f(x_n)} \frac{dz}{z\phi(z)} = -T$$

If H_n adults are harvested each cycle, the model is

$$x_{n+1} = R_n - H_n = f(x_n) - H_n$$

The Ricker stock recruitment model has the form

$$\frac{dz}{dt} = -zB_n = -\alpha x_n z, \quad z(0) = \alpha x_n$$

which has the solution

$$z = \alpha x_n e^{-\alpha x_n t}$$

so that

$$R_n = \alpha x_n e^{-\alpha T x_n}$$

which, letting $\beta = \alpha T$, becomes

$$R_n = \alpha x_n e^{-\beta x_n}$$

and the metered model is:

$$x_{n+1} = \alpha x_n e^{-\beta x_n}$$

An equilibrium or fixed point for the Ricker model would satisfy

$$\overline{x} = \alpha \overline{x} e^{-\beta \overline{x}}$$

Problem 1: Determine the fixed points assuming $\alpha > 1$.

Problem 2: Construct a spreadsheet that illustrates the equilibrium for values of β and α of your choosing.