MTH396 Spring 2013 Final Exam

Problem 1. Suppose we have a random sample $Y = (Y_1, Y_2, \ldots, Y_n)$. The simplest linear model has a single parameter for the mean of Y:

$$y = \beta + e \quad e \sim \text{ IID } N(0, \sigma_e^2)$$

1) If this model is written in the form $Y = X\beta + e$, what is the design matrix X?

2) What is the rank of X?

3) Show that

$$\hat{\beta} = (X'X)^{-1}X'Y = \overline{Y}$$

4) What is the vector of fitted values

$$X\hat{\beta}?$$

5) What are the residuals,

$$Y - X\hat{\beta}?$$

6) Show that

$$X(X'X)^{-1}X' = \frac{1}{n}J$$

where J is an $n \times n$ matrix of all ones (hint: $(X'X)^{-1})$ is a scalar, so you can write this as $(X'X)^{-1}XX')$. and what is its rank?

7) Show that

$$X(X'X)^{-1}X' \quad \text{is idempotent}$$

8) Show that

$$(I - X(X'X)^{-1}X') \quad \text{is idempotent}$$

9) Show that the vector of residuals is

$$Y - X\hat{\beta} = (I - X(X'X)^{-1}X')Y$$

10) Show that sum of the squares of the residuals is:

$$Y'AY \quad \text{with} \quad A = (I - X(X'X)^{-1}X')$$

What is the rank of A? (hint: compare the sum of the first $n-1$ columns to the n^{th} column).

11) Show that sum of the squares of the fitted values is:

$$Y'BY \quad \text{with} \quad B = X(X'X)^{-1}X'$$

What is the rank of B?

12) If Y has a multivariate normal distribution with mean vector μ and variance-covariance matrix V, the quadratic form $Y'AY$ has a chi-square distribution with rank(A) degrees of freedom and noncentrality parameter $\mu'A\mu$ if and only if AV is idempotent, that is, $(AV)^2 = AV$.

Show that if Y represents a random sample from a $N(\mu, \sigma_e^2)$, and

$$A = \frac{1}{\sigma_e^2}(X(X'X)^{-1}X')$$

then $Y'AY$ has a chi-square distribution with one degree of freedom and noncentrality parameter $n\mu^2/\sigma_e^2$. 13) Show that if Y represents a random sample from a $N(\mu, \sigma_e^2)$, and

$$B = \frac{1}{\sigma_e^2}(I - X(X'X)^{-1}X')$$

then $Y'BY$ has a chi-square distribution with $n-1$ degrees of freedom and noncentrality parameter zero (a *central* chi-square).

14) The quadratic forms $Y'AY$ and $Y'BY$ are independent if $BVA = 0$ where $V = \text{var}(Y)$. Show that for our random sample Y,

$Y'(X(X'X)^{-1}X')Y$ and $Y'(I - X(X'X)^{-1}X')Y$ are independent.

15) The ratio of two independent chi-square random variables divided by their respective degrees of freedom has an F distribution.

$$\frac{Y'AY}{Y'BY/(n-1)}$$

If the noncentrality parameter of the numerator is not zero, this is called a *noncentral* F distribution. Otherwise, it is called a *central* F distribution (or just an F distribution)

16) Show that if $\mu = 0$, the resulting F distribution has noncentrality parameter zero (this provides a test of the hypothesis that $\mu = 0$)