1. ASSIGNMENT 4

1.1. **Problem 1.** If a probability distribution depends on a single parameter θ where $\theta \in \Theta$ for some set Θ of possible parameter values, the collection of distributions taken over all possible values of θ is said to form a **one-parameter exponential family** if there exist real-valued functions $\eta, B : \Theta \to \mathbb{R}$ and $T, h : \mathbb{R}^q \to \mathbb{R}$ such that the density function $f(x;\theta)$ or probability mass function $p(x;\theta)$ of any distribution in the collection has the form

$$p(x;\theta) \text{ or } f(x;\theta) = h(x) \exp\left[\eta(\theta)T(x) - B(\theta)\right]$$

a) Show that for values of $\theta \in \Theta = (0, 1)$ and a given value of n, the binomial distributions $B(n;\theta)$

$$p(x;\theta) \;=\; \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

form a one-parameter exponential family with $q = 1$ by identifying $h(x), \eta(\theta), T(x)$, and $B(\theta)$.

b) The function $T(x)$ is called the **natural sufficient statistic** of the family. Use the Neyman factorization theorem to show that $T(x)$ is sufficient for θ in any distribution that belongs to a one-parameter exponential family.

1.2. **Problem 2.** Suppose $Y = (Y_1, \ldots, Y_n)$ is a vector of n IID random variables from a one-parameter exponential family with $q = 1$.

$$f(y_i, \theta) \;=\; h(y_i) \exp\left[\eta(\theta)T(y_i) - B(\theta)\right]$$

Show that the joint density function of Y, $f(Y;\theta)$ belongs to a one-parameter exponential family with $q = n$ by identifying $h(x), \eta(\theta), T(x)$, and $B(\theta)$.

1.3. **Problem 3.** If $y = (y_1, \ldots, y_n)$ is a random sample from a population with density function

$$f(y;\theta) = (1+\theta)y^\theta \quad 0 < 1 < 1, \quad \theta > -1$$

find the maximum likelihood estimator of θ. (Hint: It is sometimes easier to find the value of θ that maximizes the log of the likelihood function)

1.4. **Problem 4.** Let $y = (y_1, \ldots, y_n)$ be a random sample of size n from an exponential distribution with parameter θ.

a) Show that

$$\overline{y} = \frac{1}{n} \sum_{i=1}^{n}$$

is an efficient estimator for θ

b) Show that $\overline{y}$ is also the maximum likelihood estimator for θ