1. Basic Definitions

The approach we will take is sometimes called "Naive Set Theory" because we do not attempt to give a rigorous definition of a set (this is considered impossible). So we'll think of a set as a collection of objects, and consider a set S to be well-defined if, given an arbitrary object x, we can determine whether or not it is in the set.

Definition (subset)**.** *Let A and S be sets. We say that A is a **subset** of S and write $A \subseteq S$ if every element of A also belongs to S, that is,*

$$x \in A \Rightarrow x \in S$$

Equivalently, we can say

$$x \in S \quad \forall \quad x \in A$$

or

$$\nexists x \in A \quad \textit{such that} \quad x \notin S$$

Definition (equality of sets)**.** *Two sets A and B are defined to be equal if*

$$A \subseteq B \quad \textit{and} \quad B \subseteq A$$

This means every element of A belongs to B and vice-versa.

The definition is a way of stating that A and B have the same elements, without getting tangled up in a definition of elementwise equality, which could be quite tricky considering that the elements of a set can be anything.

Definition (proper subset)**.** *Let A and S be sets. We say that A is a **proper subset** of S and write $A \subset S$ if*

$$A \subseteq S \quad \textit{and} \quad A \neq S$$

Definition (null set)**.** *The **null set** or **empty set** denoted by*

$$\emptyset \quad \textit{or} \quad \{\}$$

is the set which contains no elements.

Note that $\emptyset \subseteq S$ for any set S because

$$\nexists x \in \emptyset \quad \text{such that} \quad x \notin S$$

Definition (cardinality; equivalence)**.** *The* ***cardinality*** *of a set S, denoted by $n(S)$, is the number of elements in S. Two sets A and B that have the same cardinality (that is, $n(A) = n(B)$), are said to be* ***equivalent*** *and we denote this by $A \sim B$.*

Definition (power set)**.** *The* ***power set*** *of a set S, denoted by $\mathcal{P}(S)$, is the set of all possible subsets of S (including S itself and $\emptyset$).*

A finite set S has $2^{n(S)}$ subsets.

Example. *Let $S = \{1, 3, 5\}$. Then the power set $\mathcal{P}(S)$ consists of the following $2^{n(S)} = 2^3$ subsets:*

$$\mathcal{P}(S) = \{\{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}, \emptyset\}$$

2. Set Operations

We define the following constructs:

Definition (set union)**.** *The* ***union*** *of two sets A and B, denoted by $A \cup B$, is the set consisting of all elements that belong to A, or belong to B, or belong to both A and B:*

$$A \cup B = \{x | x \in A \quad or \quad x \in B\}$$

Definition (set intersection)**.** *The* ***intersection*** *of two sets A and B, denoted by $A \cap B$, is the set consisting of all elements that belong to* ***both*** *A and B:*

$$A \cup B = \{x | x \in A \quad and \quad x \in B\}$$

Definition (set compliment)**.** *The* ***compliment*** *of a set A, denoted by A^c, is the set consisting of all elements that* ***do not*** *belong to A. Implicit in this definition is the notion of a universal set U, which may be given or may be clear from the context. Somewhat more precisely, we can say that the compliment of A relative to U, denoted by $U \setminus A$, consists of all elements of U that do not belong to A:*

$$A^c = U \setminus A = \{x | x \in U \quad and \quad x \notin A\}$$

The following useful identities, which relate these constructs, are know as the *DeMorgan Laws*:

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c$$

Definition (Cartesian product)**.** *The* ***Cartesian product*** *of a two sets A and B, denoted by $A \times B$, is the set of all possible ordered pairs having the first entry belonging to A and the second belonging to B:*

$$A \times B = \{(a, b) \mid a \in A \quad \textit{and} \quad b \in B\}$$

3. Rigorous definition of a function

Although we are all familiar with the notion of a function, the definitions one usually encouters early on are not sufficiently general for more abstract settings. Here we present an equivalent definition that possesses the necessary generality:

Definition (function)**.** *If A and B are sets, a* ***function*** *mapping A to B, denoted by $f : A \to B$, is a subset of the Cartesian product $A \times B$ with the property that every element of A appears as the first element of exactly one ordered pair in the subset.*

In keeping with the spirit of casting Mathematics in the language of set theory, we note in passing that with this definition a function is in fact a set. Denoting the subset of $A \times B$ that defines our function by f, we can write:

$$f(a) = b \quad \Leftrightarrow \quad (a, b) \in f$$

A bit of thought should convince you that this definition covers everything the earlier definitions we encouter covers.

Definition (image of a set under a function; range)**.** *If $f : A \to B$ is a function and $E \subseteq A$, the* ***image of*** *E* ***under*** *f, denoted by $f[E]$, is defined to be the set of all $b \in B$ such that $(a, b) \in f$ for some $a \in E$:*

$$f[E] = \{b \mid (a, b) \in f \quad \textit{for some} \quad a \in E\}$$

In the special case where $E = A$, $f[A]$ is called the **range** of f. A function $f : A \to B$ with the property that $f[A] = B$ is said to be **onto** or **surjective**.

Now we define the inverse image of a set under a function, which plays an important role in many areas:

Definition (inverse image of a set under a function; range). *If $f: A \to B$ is a function and $G \subseteq B$, the **inverse image of** G **under** f, denoted by $f^{-1}[G]$, is defined to be the set of all $a \in A$ such that $(a,b) \in f$ for some $b \in G$:*

$$f^{-1}[G] = \{a \mid (a,b) \in f \quad \textit{for some} \quad b \in G\}$$

A function with the property that $f^{-1}[b]$ consists of a single element of A for every b in the range $f[A]$ is said to be **one-to-one** or **injective**.

A function that is both injective and surjective is called a **bijection** or a **one-to-one correspondence**.

It is important to note that these are the only functions that have a **function inverse** defined by

$$f: A \to B,\ f^{-1}: B \to A \quad \text{such that} \quad \begin{cases} f^{-1}(f(x)) = x & \forall \quad x \in A \\ f(f^{-1}(y)) = y & \forall \quad y \in B \end{cases}$$

Be careful to distinguish between the function inverse $f^{-1}(y)$ and the inverse image $f^{-1}[G]$ for $G \subseteq B$.

The function inverse only exists for a special class of functions, but the inverse image of any $G \subseteq B$ always exists for any function $f: A \to B$.

As it turns out, inverse images have the nicest set of properties one could ask for with regard to unions, intersections, and compliments: If $f: A \to B$ is a function and $E, G \subseteq B$, the following identities hold:

$$f^{-1}[E \cup G] = f^{-1}[E] \cup f^{-1}[G]$$

$$f^{-1}[E \cap G] = f^{-1}[E] \cap f^{-1}[G]$$

$$f^{-1}[E^c] = f^{-1}[B \setminus E] = \left(f^{-1}[E]\right)^c = A \setminus f^{-1}[E]$$

In the preceding, for E^c the universal set is understood to be B, and for $f^{-1}[E]$ the universal set is A:

$$E^c = B \setminus E$$

$$(f^{-1}[E])^c = A \setminus f^{-1}[E]$$

The forward image $f[G]$ for $G \subseteq A$ does not quite have the same set of properties. While it is true that

$$f[E \cup G] = f[E] \cup f[G] \quad \text{for any} \quad E, G \subseteq A$$

the most we can say for the intersection is

$$f[E \cap G] \subseteq f[E] \cap f[G] \quad \text{for any} \quad E, G \subseteq A$$