1. Background Material from Linear Algebra

It is assumed that you are familiar with the definition of a matrix and how to multiply vectors and matrices.

A vector or matrix is said to be $n \times m$ if it has n rows and m columns.

A *vector* or *column vector* will refer to an $n \times 1$ array:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The *transpose* of a vector v, denoted by v' or v^T, is obtained by writing v as a *row vector*:

$$\text{if} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{then} \quad v' = [v_1 \; v_2 \; \cdots \; v_3]$$

In statistics texts, the prime notation is more common.

The transpose of a matrix A, denoted by A' or A^T, is obtained by interchanging the rows and columns of A. That is, the first column of A becomes the first row of A', the second column of A becomes the second row of A', and so on. For example,

$$\text{if} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \text{then} \quad A' = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

The *identity matrix* I or I_n is a square matrix having ones on the main diagonal and zeroes elsewhere:

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

For any square matrix A,

$$AI = IA = A$$

With each *square* matrix A there is associated a unique scalar called the *determinant* of A, denoted by $|A|$.

The *rank* of a matrix A, denoted by $Rank(A)$ or $r(A)$, is the number of linearly independent rows (or colunms) that A has. The number of linearly independent rows is always the same as the number of linearly independent columns.

The *trace* of an $n \times n$ matrix A, denoted by $Tr(A)$, is the sum of its diagonal entries:

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

If the determinant of a matrix A is not zero, there exists a unique *inverse* of A, denoted by A^{-1}, with the property that:

$$AA^{-1} = A^{-1}A = I$$

In general, multiplication of matrices is not commutative:

$$AB \neq BA \quad \text{(in most cases)}$$

An $n \times n$ matrix A with the property that:

$$a_{ij} = a_{ji} \quad \forall i, j \leq n$$

is called *symmetric.*

An $n \times n$ matrix A is said to be *idempotent* if

$$A^2 = A$$

If x and y are vectors with n elements, and A is an $n \times n$ matrix, an expression of the form

$$x'Ax$$

is called a *quadratic form.*

An expression of the form

$$x'Ay$$

is called a *bilinear form.*

2. The Variance-Covariance Matrix

Suppose

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

is a vector of jointly distributed random variables, the *variance-covariance matrix* V of Y is defined as:

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{bmatrix}$$

That is,

$$v_{ii} = \sigma_i^2 = Var(Y_i) \quad \text{and} \quad v_{ij} = \sigma_{ij} = Cov(Y_i, Y_j)$$

where

$$Var(Y_i) = E(Y_i^2) - [E(Y_i)]^2 \quad \text{and} \quad Cov(Y_i, Y_j) = E(Y_i Y_j) - E(Y_i)E(Y_j)$$

Note that V is symmetric since $Cov(Y_i, Y_j)$ is always equal to $Cov(Y_j, Y_i)$.

3. Linear Combinations

If

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

is a vector of random variables with expected value

$$E(Y) = \mu = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

and variance-covariance matrix

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{bmatrix}$$

Any linear combination of the elements of Y can be written in the form $t'Y$ for some vector t coefficients. If

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

is a vector of random variables with expected value

$$E(Y) = \mu = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

and variance-covariance matrix

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{bmatrix}$$

The expected value and variance of $t'Y$ is:

$$E(t'Y) = t'E(Y) = t'\mu \quad \text{and} \quad Var(t'Y) = t'Vt$$

Example. *Suppose*

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

is a vector of random variables with expected value

$$E(Y) = \mu = \begin{bmatrix} E(Y_1) \\ E(Y_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

and variance-covariance matrix

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

Find the expected value and variance of $U = 3Y_1 - 4Y_2$.

In this case,

$$t = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

so

$$E(t'Y) = t'\mu = [3 \ \ -4] \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = 3\mu_1 - 4\mu_2$$

and

$$Var(t'Y) = t'Vt = [3 \ \ -4] \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= [(3\sigma_1^2 - 4\sigma_{12}) \ (3\sigma_{12} - 4\sigma_2^2)] \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= 9\sigma_1^2 - 12\sigma_{12} - 12\sigma_{12} + 16\sigma_2^2$$

$$= 9\sigma_1^2 - 24\sigma_{12} + 16\sigma_2^2$$

More generally, a vector of linear combinations of the elements of Y can be written in the form $A'Y$ for a matrix A of coefficients. The expected value and variance-covariance matrix of $A'Y$ is:

$$E(A'Y) = A'E(Y) = A'\mu \quad \text{and} \quad V = A'VA$$

Example. *This time suppose*

$$U = \begin{bmatrix} Y_1 + Y_2 \\ Y_1 - Y_2 \end{bmatrix} = A'Y$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Then

$$E(U) = E(A'Y) = A'\mu = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \mu_1 + \mu_2 \\ \mu_1 - \mu_2 \end{bmatrix}$$

and the variance-covariance matrix of $A'Y$ is:

$$V_U = A'VA' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 + \sigma_{12} & \sigma_{12} + \sigma_2^2 \\ \sigma_1^2 - \sigma_{12} & \sigma_{12} - \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 + 2\sigma_{12} + \sigma_2^2 & \sigma_1^2 - \sigma_2^2 \\ \sigma_1^2 - \sigma_2^2 & \sigma_1^2 - 2\sigma_{12} + \sigma_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} Var(Y_1 + Y_2) & Cov(Y_1 + Y_2, Y_1 - Y_2) \\ Cov(Y_1 + Y_2, Y_1 - Y_2) & Var(Y_1 - Y_2) \end{bmatrix}$$

From this result we see that, for example,

$$Cov(Y_1 + Y_2, Y_1 - Y_2) = \sigma_1^2 - \sigma_2^2 = Var(Y_1) - Var(Y_2)$$