1. Written Assignment 5

1.1. **Problem 1.** A class of functions known as *linear operators* is defined by the following properties: If L is a linear operator, then for any X, Y in the domain of L and any real number α,

$$L(\alpha X) = \alpha[L(X)] \quad \text{and} \quad L(X+Y) = L(X) + L(Y)$$

Assuming that the "expectation operator" $E()$ has these properties, show that if X is a random variable and c is a constant,

$$E[c] = c$$

(hint: Define a function $f(X)$ that maps all values of X into c, and use the properties of linear operators)

1.2. **Problem 2.** Assuming that the "expectation operator" $E()$ is a linear operator, show that if X is a discrete random variable and $E(X) = \mu$,

a)

$$V(X) = E[(X-\mu)^2] = E(X^2) - 2\mu E(X) + [E(X)]^2$$

provided all of the sums converge.

b) Show that the above expression simplifies to:

$$V(X) = E[(X-\mu)^2] = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$$

1.3. **Problem 3.** The moment generating function of a random variable X is defined as:

$$m(t) = E(\exp(tX))$$

provided the expected value exists in some neighborhood of $t = 0$. For our purposes, we can treat t as a constant.

Use the binomial expansion formula

$$(a+b)^n = \binom{n}{n} a^n b^0 + \binom{n}{n-1} a^{n-1} b^1 + \cdots + \binom{n}{0} a^0 b^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

to show that the moment generating function of the binomial distribution with n trials and probability of success p on each trial is:

$$m(t) = (pe^t + 1 - p)^n$$

1.4. **Problem 4.** We can think of a probability distribution as a combination of a probability space $(\Omega, \mathcal{F}, \rho)$ and a random variable X defined on that probability space. For example, in the case of the binomial experiment we define a random variable X whose value is the number of successes in the outcome of the experiment on the underlying probability space corresponding to n independent Bernoulli trials with probability of success p. We say that the random variable X has a binomial distribution.

There is a 1-1 correspondence between probability distributions and moment generating functions, that is, if you know the moment generating function, you can identify the distribution and vice versa.

An important theorem states that if $X_1, X_2, \ldots, X_n$ are independent random variables with moment generating functions $m_1(t), \ldots, m_n(t)$, then the moment generating function of the random variable X defined as their sum,

$$X = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$$

is the product of the individual moment generating functions:

$$m_X(t) = E(e^{tX}) = E(\exp[t(X_1+X_2+\cdots+X_n)]) = \prod_{i=1}^{n} e^{tX_i} = \prod_{i=1}^{n} m_i(t)$$

a) Show that the moment generating function of the Bernoulli distribution with probability of success p is:

$$m(t) = pe^t + 1 - p$$

b) Use the fact that there is a 1-1 correspondence between moment generating functions and probability distributions to show that a random variable defined as sum of n independent Bernoulli trials each with probability of success p should have a binomial distribution.

1.5. **Optional Extra Credit Problem 5.** Suppose X is a discrete random variable and A is a subset of the possible values of X, that is,

$$A \subseteq X[\Omega]$$

Define the **indicator function** of A by:

$$I_A : X[\Omega] \to \{0, 1\} \quad \text{by} \quad I_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

Suppose X is a discrete random variable and c is a constant, and let

$$A = \{x : x \in X[\Omega] \text{ and } x > c\}$$

Show that

$$P(X > c) = E(I_A(X))$$