1. Written Assignment 4

1.1. **Problem 1.** An important concept in probability theory is that of *expected value*. If X is a discrete random variable defined on a probability space $(\Omega, \mathcal{F}, \rho)$, then the expected value of X, denoted by $E(X)$, is:

$$E(X) = \sum_{x \in X[\Omega]} x \cdot p(X = x)$$

(The notation $x \in X[\Omega]$ means that the sum is taken over all values of x in the image of Ω under the function X, which is equivalent to saying the sum is taken over all values in the range of X, or, in less technical terms, all values that the random variable X can assume).

Suggestion: construct a spreadsheet to solve parts b) and c) and use it in subsequent problems.

a) Suppose X is a random variable having the Bernoulli distribution with probability of success p:

$$P(X = x) = \begin{cases} p & \text{if } \ x = 1 \\ 1 - p & \text{if } \ x = 0 \end{cases}$$

What is $E(X)$?

b) Suppose X is a random variable representing the number showing when a single die is rolled. Assuming the die is not loaded, what is $E(X)$?

c) Finally, suppose X is a random variable representing the sum of the numbers showing when a pair of fair dice are thrown. What is $E(X)$?

1.2. **Problem 2.** The idea of expected value can be extended to include an arbitrary function of a random variable. If X is a discrete random variable defined on a probability space $(\Omega, \mathcal{F}, \rho)$, and $f : X \to \mathbb{R}$ is a function of X, then the expected value of $f(X)$, denoted by $E(f(x))$, is:

$$E(f(X)) = \sum_{x \in X[\Omega]} f(x) \cdot p(X = x)$$

a) Suppose X is a random variable having the Bernoulli distribution with probability of success p:

$$P(X = x) = \begin{cases} p & \text{if } \ x = 1 \\ 1 - p & \text{if } \ x = 0 \end{cases}$$

What is $E(X^2)$?

b) Suppose X is a random variable representing the number showing when a single die is rolled. Assuming the die is not loaded, what is $E(X^2)$?

c) Finally, suppose X is a random variable representing the sum of the numbers showing when a pair of fair dice are thrown. What is $E(X^2)$?

1.3. **Problem 3.** Show that if X is a discrete random variable and $E(X)$ exists, the expected deviation of X from its expected value is zero, that is,

$$E[X - E(X)] = 0$$

1.4. **Problem 4.** It is useful to have a measure that captures, in some sense, the "average" deviation of a random variable from its expected value, $X - E(X)$. In light of the results of problem 3, simply taking the expected value of this difference won't work because it always produces zero as the positive and negative deviations cancel out. One way to avoid this is to take the expected value of the *squared* deviations, which represents the average squared deviation of X from its expected value:

$$E[(X - E(X))^2]$$

This important quantity is known as the *variance* of X, and is denoted by $V(X)$ or $\mathrm{Var}(X)$.

a) Suppose X is a random variable having the Bernoulli distribution with probability of success p:

$$P(X = x) = \begin{cases} p & \text{if} \quad x = 1 \\ 1 - p & \text{if} \quad x = 0 \end{cases}$$

What is $V(X)$?

b) Suppose X is a random variable representing the number showing when a single die is rolled. Assuming the die is not loaded, what is $V(X)$?

c) Finally, suppose X is a random variable representing the sum of the numbers showing when a pair of fair dice are thrown. What is $V(X)$?