1. MTH395 Midterm

1.1. **Problem 1.** A function ϕ is *convex* on the interval (a, b) if for any $x, y \in (a, b)$ and any $\lambda \in (0, 1)$,

$$\phi[\lambda x + (1 - \lambda)y] \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$$

If ϕ is convex on an interval (a, b) and Y is a random variable that assumes only two values $y_1, y_2 \in (a, b)$, neither of which has probability zero, show that

$$\phi[E(Y)] \leq E[\phi(Y)]$$

1.2. **Problem 2.** Suppose (S, Σ, μ) is a probability triple associated with an experiment with $S = \{0, 1, 2, 3, \ldots\}$ and Y is a random variable defined on that triple. Show that

$$E(Y) = \sum_y [1 - F(y)]$$

where $F(y) = P(Y \leq y)$ is the cumulative distribution function of Y.

1.3. **Problem 3.** The *correlation* between two random variables X and Y is defined as:

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Show that if Y_1 and Y_2 are identically distributed (i.e., have the same marginal distribution) then the correlation between $Y_1 + Y_2$ and $Y_1 - Y_2$ is zero.

1.4. **Problem 4.** An operator A is said to be **idempotent** if $A(A(x)) = A(x)$. Show that expected value is idempotent in the sense that if Y is a random variable and $E(Y)$ exists, then

$$E[E(Y)] = E(Y)$$

1.5. **Problem 5.** If Y is a random vector

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

with joint moment-generating function $M(t_1, t_2)$, the moment-generating function of the random variable $X = Y_1 + Y_2$ is $M(t, t)$. Suppose Y_1 and Y_2 have joint density function

$$f(x, y) = e^{-y_1 - y_2}, \quad 0 < y_1, y_2 < \infty$$

Find:

a) The joint moment-generating function $M(t_1, t_2)$ of Y_1 and Y_2
b) The moment-generating functions of the marginal distributions of Y_1 and Y_2, and identify the marginal distributions.
c) The moment-generating function of the sum $Z = Y_1 + Y_2$.

1.6. **Problem 6.** Show that if X is a discrete random variable and $\mathrm{Var}(X) = 0$ then there exists an $a \in \mathbb{R}$ such that $P(X = a) = 1$

1.7. **Problem 7.** Show that for any random discrete random variables X and Y

$$E[(XY)^2] \leq E(X^2) \cdot E(Y^2)$$

with equality if and only if $P(aX = bY) = 1$ for some $a, b \in \mathbb{R}$ with a and b not both zero. (Hint: consider the random variable $Z = aX - bY$ and use the results of problem 6)

1.8. **Problem 8.** Let X and Y be independently distriuted Poisson random variables with parameters λ and μ, respectively. Show that:

a) $Z = X + Y$ has a Poisson distribution with mean $\lambda + \mu$
b) The conditional distribution of X, given $X + Y = n$ has a binomial distribution
c) Find the parameters of the binomial distribution in part b).