1. Assignment 4

1.1. **Problem 1.** An urn contains a red chips and b black chips. A chip is selected randomly from the urn and then the selected chip is returned to the urn and a chip of the same color is added to the urn. This process is repeated three times. If the third chip drawn is black, what is the probability that the first chip drawn was red?

1.2. **Problem 2.** Two tetrahedral dice are weighted so that the probability of a given face turning up is proportional to the number on the face. That is, for each die,

$$S_1 = S_2 = \{1, 2, 3, 4\}, \quad \mu_1(i) = \mu_2(i) = ki, \quad i = 1, 2, 3, 4$$

a) If the experiment of rolling both dice is represented by the probability triple (S, Σ, μ), what are S, Σ, and μ if the value on die 1 is independent of the value on die 2?

b) If a random variable $Y : S \to \mathbb{R}$ is defined on this probability triple in such a way that

$$Y\{(i, j)\} = i \cdot j$$

what is the value of the corresponding probability function $p(y)$ when $y = 4$?

1.3. **Problem 3.** Wallace and Gromit play a game where a fair coin is tossed. If the result is heads, Gromit pays Wallace \$1, but if the result is tails, Wallace pays Gromit \$1. The probability space for this game is the coin toss model,

$$(S, \Sigma, \mu) \quad \text{with} \quad S = \{H, T\}, \quad \Sigma = \mathcal{P}(S), \quad \text{and } \mu(\{H\}) = \mu(\{T\}) = \frac{1}{2}$$

Define a random variable

$$Y_1 : S \to \mathbb{R}$$

in such a way that Y_1 is the change in the amount of money Wallace has after the experiment is performed.

a) What is the range of this random variable, $Y_1[S]$?

b) What is the probability function $p_1(y) = P(Y_1 = y)$?

c) What is the probability triple $(S_{10}, \Sigma_{10}, \mu_{10})$ associated with the combined experiment of tossing the coin 10 times? (Assume each trial is independent for the purpose of determining μ_{10}).

d) Define a random variable

$$Y_{10} : S_{10} \to \mathbb{R}$$

in such a way Y_{10} represents the change in the amount of money Wallace has after 10 trials, that is, Y_{10} is the sum of the 10 independent random variables Y associated with the individual trials. If $p_{10}(y)$ is the probability function for Y_{10}, what is $p_{10}(2)$? (Hint: consider all points in S_{10} to be equally likely together with the fact that Wallace has to win exactly 6 tosses to have $Y_{10} = 2$).

1.4. **Problem 4.** With reference to the example where a randomly chosen subject who may or may not have a disease is given a clinical test for the disease that will be either positive or negative, the **sensitivity** of the test is defined as the conditional probability that the test is positive given that the subject has the disease:

$$s_1 = P(\text{positive}|\text{disease})$$

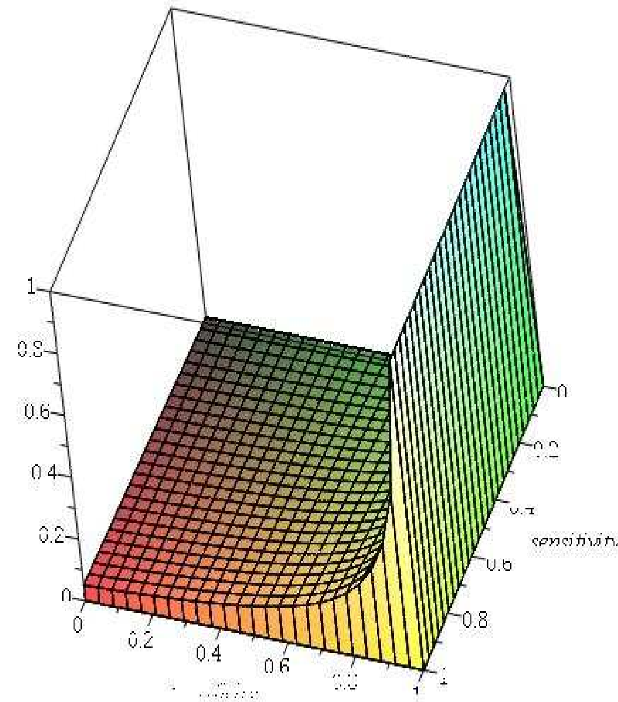
The **specificity** of the test is defined as the conditional probability of a negative result given that the subject does not have the disease:

$$s_2 = P(\text{negative}|\text{no disease})$$

Finally, the **positive predictive value** P_{pv} of the test is defined as the conditional probability that a (randomly selected) subject has the disease, given that the test result is positive.

a) Find an expression for P_{pv} as a function of P_d, the probability that a randomly selected subject has the disease, the sensitivity s_1, and the specificity s_2.

b) Given that we have no control over P_d, how can the positive predictive value be increased for a given value of s_1? (The graph of your function $P_{pv}(P_d, s_1, s_2)$ should look something like this if $P_d = 0.05$)



1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
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1
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sensitivity
0.8
1