## Contents



# VECTOR SPACES

## Preliminaries

*Definition* 1 (group). A **group** consists of:

- A set $G$
- A binary operation $+ : G \times G \to G$ with the following properties:
  $x + (y + z) = (x + y) + z \;\forall x, y, z \in G$ (associativity)
  $\exists 0 \in G$ such that $a + 0 = 0 + a = a \;\forall a \in G$ (identity)
  $\forall a \in G \;\exists\, a^{-1}$ such that $a + a^{-1} = a^{-1} + a = 0$ (inverse)

*Definition* 2 (field). A **field** consists of:

- A set $F$
- A binary operation $+ : F \times F \to F$ with the following properties:
  $x + y = y + x \;\forall x, y \in F$ (additive commutativity)
  $x + (y + z) = (x + y) + z \;\forall x, y, z \in F$ (additive associativity)
  $\exists 0 \in F$ such that $a + 0 = 0 + a = a \;\forall a \in F$ (additive identity)
  $\forall a \in F \;\exists\, a^{-1}$ such that $a + a^{-1} = a^{-1} + a = 0$ (additive inverse)
- A binary operation $: F \times F \to F$ with the following properties:
  $xy = yx \;\forall x, y \in F$ (multiplicative commutativity)
  $x(yz) = (xy)z \;\forall x, y, z \in F$ (multiplicative associativity)
  $\exists 1 \in F$ such that $a1 = 1a = a \;\forall a \in F$ (multiplicative identity)
  $\forall a \in F \setminus 0 \;\exists\, a^{-1}$ such that $aa^{-1} = a^{-1}a = 1$ (multiplicative inverse)
  $x(y + z) = xy + xz \quad \forall x, y, z \in F$ (distributive property)

## Definitions

*Definition* 3 (vector space). A **vector space** or **linear space** consists of:

- A field $F$ of elements called **scalars**
- A commutative group $V$ of elements called **vectors** with respect to a binary operation $+$
- A binary operation : $F \times V \to V$ called **scalar multiplication** that associates with each scalar $\alpha \in F$ and vector $v \in V$ a vector $\alpha v$ in such a way that:
  $1v = v \quad \forall v \in V$
  $(\alpha\beta)v = \alpha(\beta v) \quad \forall \alpha, \beta \in F,\ v \in V$
  $\alpha(v + w) = \alpha v + \alpha w \quad \forall \alpha \in F,\ v, w \in V$
  $(\alpha + \beta)v = \alpha v + \beta v \quad \forall \alpha, \beta \in F,\ v \in V$

Note that a vector space is a composite object consisting of a field, a set of 'vectors', and two operations with the specified properties. We say that $V$ is a vector space over the field $F$. With respecte to the vector addition operation, $V$ is a commutative (Abelian) group.

## Examples

**The n-tuple space $F^n$.** Let $F$ be any field and let $V$ be the set of all $n$-tuples of scalars

$$V = \{(x_1, x_2, \ldots, x_n) \ : \ x_i \in F,\ i = 1, \ldots, n\}$$

Then for $x, y \in V$, define:

$$(x + y) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad \forall x, y \in V$$

and

$$\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \quad \forall \alpha \in F,\ x \in V$$

Specific examples are $\mathbb{R}^n$ where $F = \mathbb{R}$ and $\mathbb{C}^n$ where $F = \mathbb{C}$.

**The space polynomial functions over a field $F$.** Let $F$ be a field and $V$ the set of polynomial functions of $F$, that is, the set of all functions of the form

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

where $a_0, \ldots, a_n$ are fixed scalars in $F$.

Note that if $f$ and $g$ are polynomials on $F$ and $c \in F$, then $f + g$ and $cf$ are also polynomials in $F$.

**The space of $m \times n$ matrices $F^{m\times n}$.** If $F$ is a field and $m, n \in \mathbb{N}$, let $F^{m\times n}$ be the set of all $m \times n$ matrices over $F$. Define

$$(A+B)_{ij} = A_{ij} + B_{ij}, \quad i = 1, \ldots, m, \ j = 1, \ldots, n \quad \forall A, B \in V$$

and

$$(cA)_{ij} = c(A_{ij}), \quad i = 1, \ldots, m, \ j = 1, \ldots, n \quad \forall A \in V, \ c \in F$$

**The space of functions from a set to a field.** Let $F$ be a field and $S$ a nonempty set. Let $V$ be the set of all functions from $S$ into $F$:

$$V = \{f : S \to F\}$$

If $f, g \in V$, define:

$$(f+g)(s) = f(s) + g(s) \quad \forall f, g \in V, \ s \in S$$

and

$$(cf)(s) = cf(s) \quad \forall f \in V, \ s \in S$$

We can verify that the elements of $V$ have the properties required of vectors.

First, since $f(x)$ is always an element of the field $F$, and addition in $F$ is commutative by the properties of a field, we have

$$f(s) + g(s) = g(s) + f(s) \quad \forall s \in S$$

so the functions $f+g$ and $g+f$ are the same:

$$(f+g) = f + g \quad \forall f, g \in V$$

Second, addition is $F$ is associative, so

$$f(s) + [g(s) + h(s)] = [f(s) + g(s)] + h(s) \quad \forall s \in S$$

Define the zero function as the function which assigns the zero element of $F$ (which exists by the properties of a field) to every element of $s$:

$$\vec{0} = f : S \to F \text{ such that } f(s) = 0 \quad \forall s \in S$$

Finally, for each function $f \in V$, let $(-f)$ be defined as

$$(-f)(s) = -f(s) \quad \forall f \in V, \ s \in S$$

In these arguments we use the properties of the field $F$ to establish that a particular statement is true for each element of the domain of an arbitrary element $f \in V$, and therefore holds for the functions themselves. Similar arguments can be used to show that the required properties of scalar multiplication hold.

**The space of sequences.** Let $F$ be a field and $V$ the set of sequences $\{x_n\}$ whose elements belong to $F$:

$$V = \{(x_1, x_2, \ldots) \ : \ x_i \in F \ \forall i \in \mathbb{N}\}$$

If we think of a sequence as a function whose domain is $\mathbb{N}$ we can see that this is a special case of the previous example, the space of functions from a set to a field. In this case define the vector sum as the termwise sum of the two sequences:

$$\{(x+y)_n\} = \{x_n\} + \{y_n\} \quad \forall x, y \in V$$

and the scalar product is:

$$\alpha x = \alpha\{x_n\} = \{\alpha x_n\} \quad \forall \alpha \in F, \ x \in V$$

**The space of real-valued functions on** $[-1, 1]$**.** This is another special case of a space of functions from a set $S = [-1, 1]$ to a field $F = \mathbb{R}$.

$$V = \{f : [-1, 1] \to \mathbb{R}\} \quad F = \mathbb{R}$$

If $f, g \in V$, define:

$$(f + g)(s) = f(s) + g(s) \quad \forall f, g \in V, \ s \in [-1, 1]$$

and

$$(\alpha f)(s) = \alpha f(s) \quad \forall f \in V, \ \alpha \in \mathbb{R}$$

## 1. Norms

*Definition* 4 (norm). A nonnegative real-valued function $\| \ \| : V \to \mathbb{R}$ is called a **norm** if:

- $\|v\| \geq 0$ and $\|v\| = 0 \ \Leftrightarrow \ v = \vec{0}$
- $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality)
- $\|\alpha v\| = |\alpha| \, \|x\| \quad \forall \alpha \in F, \ v \in V$