MTH362 Spring 2012

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CHAPTER 1

Preliminaries and Definitions

Definition 1.0.1 (binary operation). *A **binary operation** on a set S is a function from $S \times S$ into S.*

Examples of binary operations:

- $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *Addition of natural numbers*
- $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *Multiplication of natural numbers*

Definition 1.0.2 (group). *A **group** consists of:*

- *A set G*
- *A binary operation $+ : G \times G \to G$ with the following properties:*

$$x + (y + z) = (x + y) + z \ \forall x, y, z \in G \qquad \textit{(associativity)}$$
$$\exists 0 \in G \ \textit{ such that } a + 0 = 0 + a = a \ \forall a \in G \qquad \textit{(identity)}$$
$$\forall a \in G \ \exists \, a^{-1} \ \textit{ such that } \ a + a^{-1} = a^{-1} + a = 0 \qquad \textit{(inverse)}$$

Definition 1.0.3 (field). *A **field** consists of:*

- *A set F*
- *A binary operation $+ : F \times F \to F$ with the following properties:*

$$x + y = y + x \ \forall x, y \in F \qquad \textit{(additive commutativity)}$$
$$x + (y + z) = (x + y) + z \ \forall x, y, z \in F \qquad \textit{(additive associativity)}$$
$$\exists 0 \in F \ \textit{ such that } a + 0 = 0 + a = a \ \forall a \in F \qquad \textit{(additive identity)}$$
$$\forall a \in F \ \exists \, a^{-1} \ \textit{ such that } \ a + a^{-1} = a^{-1} + a = 0 \qquad \textit{(additive inverse)}$$

- *A binary operation $: F \times F \to F$ with the following properties:*

$$xy = yx \ \forall x, y \in F \qquad \textit{(multiplicative commutativity)}$$
$$x(yz) = (xy)z \ \forall x, y, z \in F \qquad \textit{(multiplicative associativity)}$$
$$\exists 1 \in F \ \textit{ such that } a1 = 1a = a \ \forall a \in F \qquad \textit{(multiplicative identity)}$$
$$\forall a \in F \setminus 0 \ \exists \, a^{-1} \ \textit{ such that } \ aa^{-1} = a^{-1}a = 1 \qquad \textit{(multiplicative inverse)}$$
$$x(y + z) = xy + xz \quad \forall x, y, z \in F \qquad \textit{(distributive property)}$$

Definition 1.0.4 (vector space). *A **vector space** or **linear space** consists of:*

- *A field F of elements called **scalars***
- *A commutative group V of elements called **vectors** with respect to a binary operation $+$*

- *A binary operation* $: F \times V \to V$ *called* ***scalar multiplication*** *that associates with each scalar* $\alpha \in F$ *and vector* $v \in V$ *a vector* αv *in such a way that:*
  $1v = v \quad \forall v \in V$
  $(\alpha\beta)v = \alpha(\beta v) \quad \forall \alpha, \beta \in F,\ v \in V$
  $\alpha(v + w) = \alpha v + \alpha w \quad \forall \alpha \in F,\ v, w \in V$
  $(\alpha + \beta)v = \alpha v + \beta v \quad \forall \alpha, \beta \in F,\ v \in V$

DEFINITION 1.0.5 (norm). *A nonnegative real-valued function* $\| \ \| : V \to \mathbb{R}$ *is called a* ***norm*** *if:*

- $\|v\| \geq 0$ *and* $\|v\| = 0 \Leftrightarrow v = \vec{0}$
- $\|v + w\| \leq \|v\| + \|w\|$ *(triangle inequality)*
- $\|\alpha v\| = |\alpha| \, \|x\| \quad \forall \alpha \in F,\ v \in V$

DEFINITION 1.0.6 (normed linear space). *A linear space* V *together with a norm* $\| \cdot \|$*, denoted by the pair* $(V, \| \cdot \|)$*, is called a* ***normed linear space***

DEFINITION 1.0.7 (inner product). *Let the field* F *be either* $\mathbb{R}$ *or* $\mathbb{C}$ *and a set* V *of vectors which together with* F *form a vector space. An* ***inner product*** *on* V *is a map*

$$\cdot : V \times V \to \mathbb{F}$$

with the following properties:

$$\begin{array}{ll} (u + v) \cdot w = u \cdot w \ + \ v \cdot w & \forall u, v, w \in V \\ (\alpha u) \cdot v = \alpha(u \cdot v) & \forall \alpha \in F,\ u, v \in V \\ u \cdot v = (\overline{v \cdot u}) & \forall u, v \in V \\ u \cdot u \geq 0 & \forall u \in V \text{ with equality when } u = \vec{0} \end{array}$$

If the underlying field is $\mathbb{R}$*, the fourth condition can be replaced by*

$u \cdot v = v \cdot u \quad \forall u, v \in V$

since a real number is its own conjugate. In this case, the condition just says the inner product is commutative.

DEFINITION 1.0.8 (metric). *A* ***metric*** *on a set* S *is a function*

$$\rho : S \times S \to \mathbb{R}$$

where ρ *has the following three properties for any* $x, y, z \in S$*:*

$$\begin{aligned}&\rho(x,y) \geq 0 \;\; \textit{and} \; \rho(x,y) = 0 \Leftrightarrow x = y\\&\rho(x,y) = \rho(y,x)\\&\rho(x,y) \leq \rho(x,z) + \rho(z,y)\end{aligned}$$

Definition 1.0.9 (metric space). *A* ***metric space*** *is a pair* $\{S, \rho\}$ *where* S *is a set and* ρ *is a metric defined on* S*.*

Definition 1.0.10 (topology). *A* ***topology*** *is a set* X *and a collection* $\mathcal{J}$ *of subsets of* X *having the following properties:*

- $\emptyset$ *and* X *are in* $\mathcal{J}$
- *The union of any subcollection of elements of* $\mathcal{J}$ *belongs to* $\mathbb{J}$
- *The intersection of any finite subcollection of* $\mathcal{J}$ *belongs to* $\mathcal{J}$

CHAPTER 2

Euclidean Spaces $\mathbb{R}^n$

2.1. Algebraic Structure

DEFINITION 2.1.1 (Euclidean space). *For any natural number n, the n-fold Cartesian product of $\mathbb{R}$ with itself is called a Euclidean space and denoted by the symbol $\mathbb{R}^n$.*

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, \quad 1 \leq i \leq n\}$$

DEFINITION 2.1.2 (vector sum in Euclidean space). *For any $x, y \in \mathbb{R}^n$, define*

$$+ : R^n \times \mathbb{R}^n \to \mathbb{R}^n \quad by \quad x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

DEFINITION 2.1.3 (scalar product in Euclidean space). *For any $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, define*

$$: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \quad by \quad \alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$$

DEFINITION 2.1.4 (inner product in Euclidean space). *For any $x, y \in \mathbb{R}^n$, define*

$$\cdot : R^n \times \mathbb{R}^n \to \mathbb{R} \quad by \quad x \cdot y = (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

DEFINITION 2.1.5 (cross product in $\mathbb{R}^3$). *For any $x, y \in \mathbb{R}^3$, define*

$$\times : R^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \quad by \quad x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

DEFINITION 2.1.6 (norms in Euclidean space). *For any $x \in \mathbb{R}^n$, define*

$$\|x\| : R^n \to \mathbb{R} \quad by \quad \|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$\|x\|_1 : R^n \to \mathbb{R} \quad by \quad \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty : R^n \to \mathbb{R} \quad by \quad \|x\|_\infty ;= \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

DEFINITION 2.1.7 (Euclidean distance). *For any $x, y \in \mathbb{R}^n$, define*

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \quad by \quad d(x,y) \ = \ \|x - y\|$$

THEOREM 2.1.1. $\mathbb{R}^n$ *is a vector space.*

PROOF. Part 1. First we need to show that $\mathbb{R}^n$ with the usual definition of a sum in terms of componentwise addition,

$$x + y \ = \ (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

is a commutative group. To show this, we need to show that:

- $\mathbb{R}^n$ contains an identity element $\overrightarrow{0}$ such that $v + \overrightarrow{0} = v$ for all $v \in \mathbb{R}^n$
- $\mathbb{R}^n$ contains an inverse element $-v$ such that $v + (-v) = \overrightarrow{0}$ for all $v \in \mathbb{R}^n$
- Addition in $\mathbb{R}^n$ is associative: $u + (v + w) = (u + v) + w$
- Addition in $\mathbb{R}^n$ is commutative: $u + v = v + u$

First, define

$$\overrightarrow{0} \ = \ (0, 0, \ldots, 0)$$

which is the element of $\mathbb{R}^n$ with every component zero. Then for any $v \in \mathbb{R}^n$,

$$v + \overrightarrow{0} \ = \ (v_1 + 0, v_2 + 0, \ldots, v_n + 0)$$

but for any real number v_j, $v_j + 0 = v_j$, so

$$v + \overrightarrow{0} \ = \ (v_1 + 0, v_2 + 0, \ldots, v_n + 0) \ = \ (v_1, v_2, \ldots, v_n) \ = \ v$$

Next, define

$$-v \ = \ (-v_1, -v_2, \ldots, -v_n)$$

Then for any $v \in \mathbb{R}^n$,

$$v+(-v) \ = \ (v_1+(-v_1), v_2+(-v_2), \ldots, v_n+(-v_n)) \ = \ (0, \ldots, 0) \ = \ \overrightarrow{0}$$

Now, establish that addition is associative. Let $u, v, w \in \mathbb{R}^n$. Then

$$u + (v + w) \ = \ (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n))$$

because addition in $\mathbb{R}$ is associative, we can write this as

$$u+(v+w) \;=\; ((u_1+v_1)+w_1, (u_2+v_2)+w_2, \dots, (u_n+v_n)+w_n)) \;=\; (u+v)+w$$

This establishes that $\mathbb{R}^n$ is a group. However, we need it to be a commutative group, so we have to show that for any $u, v \in \mathbb{R}^n$,

$$u + v \;=\; v + u$$

By definition,

$$u + v \;=\; u_1 + v_1, u_2 + v_2, \dots, u_n + v_n$$

Because addition in $\mathbb{R}$ is commutative, we can write

$$u+v \;=\; (u_1+v_1, u_2+v_2, \dots, u_n+v_n) \;=\; (v_1+u_1, v_2+u_2, \dots, v_n+u_n)$$
$$= v + u$$

For the field component, we will use $\mathbb{R}$, omitting the proof that $\mathbb{R}$ is a field.

Finally, we have to define multiplication of a vector by a scalar, which has to satisfy:

$1v = v \quad \forall v \in V$
$(\alpha\beta)v = \alpha(\beta v) \quad \forall \alpha, \beta \in F,\ v \in V$
$\alpha(v + w) = \alpha v + \alpha w \quad \forall \alpha \in F,\ v, w \in V$
$(\alpha + \beta)v = \alpha v + \beta v \quad \forall \alpha, \beta \in F,\ v \in V$

For any scalar α and vector v, define

$$\alpha v \;=\; (\alpha v_1, \alpha v_2, \dots, \alpha v_n)$$

Then if 1 is the unit element of the field of scalars,

$$1v \;=\; (1v_1, 1v_2, \dots, 1v_n) \;=\; (v_1, \dots, v_n) \;=\; v$$

If $\alpha, \beta \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then

$$(\alpha\beta)v \;=\; (\alpha\beta v_1, \alpha\beta v_2, \dots, \alpha\beta v_n)$$
$$=\; (\alpha(\beta v_1), \alpha(\beta v_2), \dots, \alpha(\beta v_n)) \;=\; \alpha(\beta v)$$

If $\alpha \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$, then

$$\alpha(v + w) \;=\; \alpha(v_1 + w_1, v_2 + 2_2, \dots, v_n + w_n)$$
$$=\; (\alpha v_1 + \alpha w_1, \alpha v_2 + \alpha w_2, \dots, \alpha v_n + \alpha w_n)$$
$$=\; (\alpha v_1, \cdots, \alpha v_n) + (\alpha w_1, \dots, \alpha w_n) \;=\; \alpha v + \alpha w$$

Finally, if $\alpha, \beta \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then

$$(\alpha + \beta)v+ \;=\; (\alpha + \beta)v_1, (\alpha + \beta)v_2, \dots, (\alpha + \beta)v_n)$$
$$=\; (\alpha v_1 + \cdots, \alpha v_n) + \beta(v_1 + \cdots + \beta v_n) \;=\; \alpha v + \beta v$$

□

THEOREM 2.1.2. *" $\cdot$ " is an inner product.*

PROOF. We need to show that the dot product on $\mathbb{R}^n$ defined by

$$x \cdot y \;=\; x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

is an inner product.

First we need to show that for $u, v, w \in \mathbb{R}^n$,

$$(u+v)\cdot w \;=\; u\cdot w + v \cdot w$$

By the definition of vector addition in $\mathbb{R}^n$,

$$u + v \;=\; (u_1+v_1, u_2+v_2, \ldots, u_n+v_n)$$

so, by the definition of the dot product,

$$\begin{aligned}(u+v)\cdot w \;&=\; ((u_1+v_1)w_1 + (u_2+v_2)w_2 + \cdots + (u_n+v_n)w_n \\ &=\; ((u_1w_1 + v_1w_1) + (u_2w_2 + v_2w_2) + \cdots + (u_nw_n + v_nw_n) \\ &=\; ((u_1w_1 + v_1w_1) + (u_2w_2 + v_2w_2) + \cdots + (u_nw_n + v_nw_n) \\ &=\; u\cdot w + v\cdot w\end{aligned}$$

Next, we need to show that for $u, v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$(\alpha u)\cdot v \;=\; \alpha(u \cdot v)$$

$$= (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)\cdot(v_1, v_2, \ldots, v_n)$$

$$= (\alpha u_1v_1 + \alpha u_2v_2 + \cdots + \alpha u_nv_n)$$

$$= \alpha(u_1v_1 + u_2v_2 + \cdots + u_nv_n) \;=\; \alpha(u\cdot v)$$

Next, we need to show that for $u, v \in \mathbb{R}^n$,

$$u \cdot v \;=\; v \cdot u$$

$$u\cdot v \;=\; (u_1v_1 + u_2v_2 + \cdots + u_nv_n)$$

by the commutativity of real multiplication, we can write this as

$$=\; (v_1u_1 + v_2u_2 + \cdots + v_nu_n) \;=\; v\cdot u$$

Finally, we need to show that for $u \in \mathbb{R}^n$,

$$u\cdot \geq 0 \quad \text{with equality only when} \quad u \;=\; \overrightarrow{0}$$

By definition,

$$u\cdot u \;=\; u_1^2 + u_2^2 + cdots + u_n^2$$

which cannot be negative since it is a sum of squared real numbers, all of which are nonnegative.

Furthermore, it can be zero only if $u_1^2 = u_2^2 = \cdots = u_n^2 = 0$ which can only happen if $u_1 = u_2 = \cdots = u_n = 0$, which makes $u = \overrightarrow{0}$. □

THEOREM 2.1.3. $\|\cdot\|$ *is a norm.*

PROOF. We need to show that:

- 1. $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$
- 2. $\|x + w\| \leq \|x\| + \|w\|$
- 3. $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in F, x \in X$

Part 1. By definition,

$$\|x\|^2 \;=\; \sum_{i=1}^{n} x_i^2 \geq 0$$

because each x_i^2 is greater than or equal to zero. since all quantities are nonnegative, taking square roots gives

$$\|x\| \geq 0$$

Next, suppose

$$\|x\|^2 \;=\; \sum_{i=1}^{n} x_i^2 = 0$$

Since all x_i^2 are greater than or equal to zero, we can only have equality if all of the x_i are zero. Finally, suppose $x = \overrightarrow{0}$. Then

$$\|x\|^2 \;=\; sum_{i=1}^{n} 0^2 \;=\; 0$$

so $\|x\| = 0$. Part 2. By definition,

$$\|x + y\|^2 \;=\; \sum_{i=1}^{n} (x_i + y_i)^2 \;=\; \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2$$

but

$$\sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \;\leq\; \sum_{i=1}^{n} x_i^2 + 2\left|\sum_{i=1}^{n} x_i y_i\right| + \sum_{i=1}^{n} y_i^2$$

By the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{n} x_i^2+2\left|\sum_{i=1}^{n} x_iy_i\right|+\sum_{i=1}^{n} y_i^2 \;\leq\; \sum_{i=1}^{n} x_i^2+2\|x\|\|y\|+\sum_{i=1}^{n} y_i^2 \;=\; (\|x\|+\|y\|)^2$$

so

$$\|x+y\|^2 \;\leq\; (\|x\|+\|y\|)^2$$

since all quantities are positive, we can take square roots on both sides to get

$$\|x+y\| \;\leq\; \|x\|+\|y\|$$

Part 3.

$$\begin{aligned}\|\alpha x\| &= \sqrt{|\alpha x_1|^2+|\alpha x_2|^2+\ldots+|\alpha x_n|^2}\\ &= \sqrt{\alpha^2|x_1|^2+\alpha^2|x_2|^2+\ldots+\alpha^2|x_n|^2}\\ &= \sqrt{\alpha^2(|x_1|^2+|x_2|^2+\ldots+|x_n|^2)}\\ &= \alpha\sqrt{\alpha(|x_1|^2+|x_2|^2+\ldots+|x_n|^2)}\\ &= |\alpha|\|x\|\end{aligned}$$

□

THEOREM 2.1.4. $\|\cdot\|_1$ *is a norm.*

PROOF. We need to show that the following three statements are true for all $\alpha \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$:

- $\|v\|_1 \geq 0 \quad \text{with} \quad |v\|_1 = 0 \Leftrightarrow v = \overrightarrow{0}$
- $\|v+w\|_1 \leq \|v\|_1 + \|w\|_1$
- $\|\alpha v\|_1 = |\alpha|\|v\|_1$

Suppose $v \in \mathbb{R}^n$. Then

$$\|v\|_1 \;=\; |v_1|+|v_2|+\cdots+|v_n| \quad \text{with each } |v_i| \geq 0$$

Since each term is greater than or equal to zero, the sum $\|v\|_1$ must also be greater than or equal to zero.

Now consider

$$\|v+w\|_1 \;=\; |v_1+w_1|+|v_2+w_2|+\cdots+|v_n+w_n|$$

$$\leq\; |v_1|+|w_1|+|v_2|+|w_2|+\cdots+|v_n|+|w_n| \;=\; \|v\|_1+\|w\|_1$$

Finally,

$$\|\alpha v\|_1 \;=\; |\alpha v_1|+|\alpha v_2|+\cdots+|\alpha v_n|$$

By the properties of absolute values, this is:

$$= |\alpha||v_1| + |\alpha||v_2| + \cdots + |\alpha||v_n| = |\alpha|(|v_1| + \cdots + |v_n|) = |\alpha|\|v\|_1$$

□

THEOREM 2.1.5. $\|\cdot\|_\infty$ *is a norm.*

THEOREM 2.1.6. $d(\cdot,\cdot)$ *is a metric.*

PROOF. By definition, a metric is a function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} &d(x,y) \geq 0 \text{ and } d(x,y) = 0 \Leftrightarrow x = y \\ &d(x,y) = d(y,x) \\ &d(x,y) \leq d(x,z) + d(z,y) \end{aligned}$$

Part 1: $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$.

We know

$$\|x_i - y_i\| = \sqrt{|x_i - y_i|^2}$$

This is greater than or equal to zero by the definition of absolute value. So, the sum

$$d(x,y) = \|x_i - y_i\| = \sqrt{|x_1 - y_1| + \cdots + |x_n - y_n|}$$

is greater than or equal to zero.

Part 2: $d(x,y) = d(y,x)$

Suppose not. Then

$$|x_1 - y_1| + \cdots + |x_n - y_n| \neq |y_1 - x_1| + \cdots + |y_n - x_n|$$

Let $c_1 = x_i - y_i$. Then

$$d(x,y) = |c_1| + |c_2| + \cdots + |c_n|$$

and $-c_1 = y_i - x_i$, and by substitution,

$$|c_1| + \cdots + |c_n| \neq |-c_1| + \cdots + |-c_n|$$

which is a contradiction since $|c_i| = |-c_i|$ for every i, $1 \leq i \leq n$. □

THEOREM 2.1.7 (Cauchy-Schwarz inequality). *For any* $x, y \in \mathbb{R}^n$,

$$|x \cdot y| \leq \|x\| \, \|y\|$$

PROOF.

$$(x - ty) \cdot (x - ty) = \|x - ty\| \geq 0$$
$$x \cdot x - 2tx \cdot y + t^2 y \cdot y \geq 0$$
$$\|x\|^2 - 2t(x \cdot y) + t^2 \|y\|^2 \geq 0$$

Let $t = \frac{(x \cdot y)}{\|y\|^2}$

$$\|x\|^2 - \frac{2(x \cdot y)^2}{\|y\|^2} + \frac{(x \cdot y)^2}{\|y\|^4} \|y\|^2 \geq 0$$
$$\|x\|^2 - \frac{(x \cdot y)^2}{\|y\|^2} \geq 0$$
$$\|x\|^2 \geq \frac{(x \cdot y)^2}{\|y\|^2}$$
$$\|x\|^2 \|y\|^2 \geq (x \cdot y)^2$$
$$\|x\| \|y\| \geq |x \cdot y|$$

□

THEOREM 2.1.8. *For any* $x \in \mathbb{R}^n$,

$$\|x\|_\infty \leq \|x\| \leq \sqrt{n} \|x\|_\infty$$

PROOF. By definition,

$$\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$$

so

$$\|x\|_\infty^2 \;=\; \max(|x_1|^2, |x_2|^2, \ldots, |x_n|^2)$$

and

$$\|x\|^2 \;=\; |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq \max(|x_1|^2, |x_2|^2, \ldots, |x_n|^2) \;=\; \|x\|_\infty^2$$

Since $\|x\|$ and $\|x\|_\infty$ are both nonnegative, we can take square roots of both terms and the inequality still holds:

$$\|x\|_\infty \leq \|x\|$$

Now consider

$$\|x\|^2 = (|x_1|^2 + \cdots + |x_n|^2) \leq n \max(|x_1^2 + \cdots + |x_n|^2) \;=\; n \|x\|_\infty^2$$

Since all quantities are nonnegative, we can write:

$$\|x\|^2 \leq \sqrt{n}\|x\|_\infty$$

and so

$$\|x\|_\infty \leq \|x\| \leq \sqrt{n}\|x\|_\infty$$

□

THEOREM 2.1.9. *For any $x \in \mathbb{R}^n$,*

$$\|x\| \leq \|x\|_1 \leq \sqrt{n}\|x\|$$

PROOF. **Part 1.**
By definition,

$$||x|| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$
$$||x||_1 = |x_1| + \cdots + |x_n|$$

Squaring each norm:

$$||x||^2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}^2 = |x_1|^2 + \cdots + |x_n|^2$$

$$||x||_1^2 = (|x_1| + \cdots + |x_n|)^2 = |x_1|^2 + \cdots + |x_n|^2 + 2 \cdot \sum |x_i||x_j| \quad \text{where } 1 < i < j < n$$

By definition of absolute values, we know that $2 \cdot \sum |x_i||x_j|$ will be greater than or equal to 0. Therefore, we can conclude:

$$|x_1|^2 + \cdots + |x_n|^2 \leq |x_1|^2 + \cdots + |x_n|^2 \ + \ 2 \cdot \sum |x_i||x_j|$$

Implying that: $||x||^2 \leq ||x||_1^2$. Taking the square root: $||x|| \leq ||x||_1$

Part 2.
Multiplying the squared Euclidean norm:

$$n\,||x||^2 = n\sqrt{|x_1|^2 + \cdots + |x_n|^2}^2 = n(|x_1|^2 + \cdots + |x_n|^2) = n \sum |x_i|^2 \quad \text{for } i = 1, \ldots, n$$

From Part 1, we say the $\ell 1$ norm squared as:

$$||x||_1^2 = (|x_1| + \cdots + |x_n|)^2 = \sum |x_i|^2 + 2 \cdot \sum |x_i||x_j| \quad \text{where } 1 < i < j < n$$

Subtracting the two norms:

$$n\,||x||^2 - ||x||_1 = n \sum |x_i|^2 - \left(\sum |x_i|^2 \ + \ 2 \cdot \sum |x_i||x_j|\right)$$

Combining like-terms:

$$\left(\sum n|x_i|^2 - |x_i|^2\right) \ + \ 2 \cdot \sum |x_i||x_j|$$

$$= \sum (n-1)|x_i|^2 \ + \ 2 \cdot \sum |x_i||x_j| = (n-1) \sum |x_i|^2 \ + \ 2 \cdot \sum |x_i||x_j|$$

Substituting in $\ell 1$ norm squared:

$$= (n-1)\,||x||_1^2$$

□

THEOREM 2.1.10. *For any $x, y \in \mathbb{R}^n$,*

$$\|x - y\| \geq \|x\| - \|y\|$$

PROOF. By definition,

$$\|x - y\|^2 = \sum_{i=1}^{n} (x_i - y_i)^2 = \sum_{i=1}^{n} x_i^2 - 2 \sum_{i=1^n} x_i y_i + \sum_{i=1}^{n} y_i^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2(x \cdot y) + \sum_{i=1}^{n} y_i^2$$

Since $|x \cdot y| \geq x \cdot y$,

$$\sum_{i=1}^{n} x_i^2 - 2(x \cdot y) + \sum_{i=1}^{n} y_i^2 \geq \sum_{i=1}^{n} x_i^2 - 2|x \cdot y| + \sum_{i=1}^{n} y_i^2$$

Using the Cauchy-Schwartz inequality, we can write

$$\sum_{i=1}^{n} x_i^2 - 2|x \cdot y| + \sum_{i=1}^{n} y_i^2 \geq \sum_{i=1}^{n} x_i^2 - 2\|x\|\|y\| + \sum_{i=1}^{n} y_i^2 = (\|x\| - \|y\|)^2$$

from which we can write

$$\|x - y\|^2 \geq (\|x\| - \|y\|)^2$$

which, since $\|x - y\| \geq 0$, implies

$$\|x - y\| \geq \|x\| - \|y\|$$

□

2.2. The Usual Topology of $\mathbb{R}^n$

Definition 2.2.1 (open ball). *For any $r > 0$ and $a \in \mathbb{R}^n$, the open ball centered at a with radius r is the set of points*

$$B_r(a) \;=\; \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

Definition 2.2.2 (closed ball). *For any $r > 0$ and $a \in \mathbb{R}^n$, the closed ball centered at a with radius r is the set of points*

$$B_r(a) \;=\; \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$$

Definition 2.2.3 (open set). *A subset O of $\mathbb{R}^n$ is said to be open if and only if for every $a \in O$, there is an $\epsilon > 0$ such that*

$$B_\epsilon(a) \subseteq O$$

Definition 2.2.4 (closed set). *A subset F of $\mathbb{R}^n$ is said to be closed if and only if*

$$F^c \;=\; \mathbb{R} \setminus F \quad \textit{is open}$$

that is, if and only if its compliment F^c is open.

Definition 2.2.5 (interior). *If E is a subset of $\mathbb{R}^n$, the interior of E is the set*

$$E^\circ \;=\; \bigcup \{V : V \subseteq E \quad \textit{and } V \textit{ is open}\}$$

that is, E° is the union of all open subsets of E.

Definition 2.2.6 (closure). *If E is a subset of $\mathbb{R}^n$, the closure of E is the set*

$$\overline{E} \;=\; \bigcap \{F : F \supseteq E \quad \textit{and } F \textit{ is closed}\}$$

that is, $\overline{E}$ is the intersection of all closed sets that contain E.

Definition 2.2.7 (boundary). *If E is a subset of $\mathbb{R}^n$, the boundary of E is the set*

$$\partial E \;=\; \{x \in \mathbb{R}^n : \textit{ for all } r > 0, \quad B_r(x) \cap E \neq \emptyset \quad \textit{and} \quad B_r(x) \cap E^c \neq \emptyset\}$$

THEOREM 2.2.1. *Suppose $a \in \mathbb{R}^n$ and $r > 0$. Let x be and arbitrary element of $B_r(a)$. Then there exists an $\epsilon > 0$ such that*

$$B_\epsilon(x) \subseteq B_r(a)$$

THEOREM 2.2.2. *Suppose $a \in \mathbb{R}^n$. Then the singleton set*

$$\{a\} \quad \text{is closed}$$

PROOF. Let F be the Singleton set containing a. The only sequence in F is $\{a, a, a, a, a...\}$, the constant sequence where every element is a. Since $lim_{n\to\infty} k_n = a \in$ F, so F contains its limit points. By theorem 3.1.15, F is closed. □

THEOREM 2.2.3. *The empty set $\emptyset$ is both open and closed.*

PROOF. Clearly for any $x \in \mathbb{R}^n$, there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathbb{R}^n$, since this statement is true for any $\epsilon > 0$. So $\mathbb{R}^n$ is open. By definition its compliment, the empty set, is closed. Now consider $\emptyset$. $\emptyset$ contains no elements, so we can say that the condition that every $x \in \emptyset$ is the center of an open ball contained in $\emptyset$ is true vacuously. □

THEOREM 2.2.4. *Considered as a set, $\mathbb{R}^n$ is both open and closed.*

PROOF. We have previously established that the empty set is open, so its compliment $\mathbb{R}^n$ is closed. Furthermore, if $x \in \mathbb{R}^n$, for any $\epsilon > 0$, $B_\epsilon(x) \subseteq \mathbb{R}^n$, so $\mathbb{R}^n$ is open. □

THEOREM 2.2.5. *The collection of open sets as defined above is a topology on $\mathbb{R}^n$*

PROOF. We need to show that the collection of sets $\mathcal{T}$ satisfying the definition of an open set form a topology, that is,

- $\mathbb{R}^n$ and $\emptyset$ are open
- Arbitrary unions of open sets are open
- Finite intersections of open sets are open

From theorems 2.3 and 2.4, $\mathbb{R}^n$ and $\emptyset$ are open. Now suppose $O_\alpha, \alpha \in A$ is a collection of open subsets of $\mathbb{R}^n$ indexed by A, and let

$$O = \bigcup_{\alpha \in A} O_\alpha$$

Then for each $x \in O$, $x \in O_\alpha$ for some $\alpha \in A$. By hypothesis, O_α is open, so there is an $\epsilon > 0$ such that

$$B_\epsilon(x) \subseteq O_\alpha$$

but $O_\alpha \subseteq O$, so we have

$$B_\epsilon(x) \subseteq O_\alpha \subseteq O$$

Since x was arbitrarily chosen, we can find such an ϵ for any $x \in O$, so O is open.

Finally, suppose O_i, $1 \leq i \leq n$ is a finite collection of open subsets of $\mathbb{R}^n$, and let

$$E = \bigcap_{i=1}^{n} O_i$$

Suppose $x \in E$. Then $x \in O_i$, for each $1 \leq i \leq n$. Since each O_i is open, there is an ϵ_i for each of them with the property that

$$B_{\epsilon_i}(x) \subseteq O_i$$

Let $\epsilon = \min(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$. Then $B_\epsilon(x)$ is contained in each of the O_i,

$$B_\epsilon(x) \subseteq O_i, \quad 1 \leq i \leq n$$

and therefore $B_\epsilon(x) \subseteq E$. Since x was arbitrarily chosen, we can find such an ϵ for any $x \in E$, so by definition E is open. □

THEOREM 2.2.6 (8.32i). *Suppose $E \subseteq R^n$. Then*

$$E^\circ \subseteq E \subseteq \overline{E}$$

PROOF. Part I: $E^o \subseteq E$

Let $x \in E^o$. We need to show $x \in E$. By defintion E^o is the union of all open subsets of E. By hypothesis, $x \in E^o$, so x belongs to at least one open subset of O_x of E. Since $x \in O_x \subseteq E$, then $x \in E$. Because x was arbitrary, every $x \in E^o$ belongs to E, so $E^o \subseteq E$. □

PROOF. Part II: $E \subseteq \overline{E}$

Now suppose $x \in E$. Let F_α be a closed set that contains E. Then $x \in E$ and $E \subseteq F_\alpha$ implies $x \in F_\alpha$. Since F_α was arbitrarily chosen, x belongs to every closed set F that contains E. So x belongs to every closed set that contains E, and therefore to their intersection, $\overline{E}$. Since x was arbitrary, every element of E belongs to $\overline{E}$, so $E \subseteq \overline{E}$.

□

THEOREM 2.2.7 (8.32ii). *Suppose $E \subseteq R^n$, V is open, and $V \subseteq E$. Then*

$$V \subseteq E^\circ$$

THEOREM 2.2.8 (8.32iii). *If $E \subseteq R^n$, F is closed, and $F \supseteq E$. Then*

$$F \supseteq \overline{E}$$

PROOF. Let x be an element of $\bar{E}$. By definition, x belongs to the intersection of all closed sets that contain E. If x belongs to the intersection, it belongs to every set in the intersection, ie, every closed set that contains E. Therefore $x \in F$ since x was arbitrary, every element of $\bar{E}$ is in F and $E \subseteq F$

□

THEOREM 2.2.9 (8.36). *Let $E \subseteq R^n$. Then*

$$\partial E \;=\; \overline{E} \setminus E^\circ$$

THEOREM 2.2.10 (8.37i). *Let $A, B \subseteq R^n$. Then*

$$(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$$

PROOF. Let $x \in A^o \cup B^o$. Then either $x \in O_A \subseteq A$ or $x \in O_B \subseteq B$. In the first case, $O_A \subseteq A \subseteq A \cup B$ so x belongs to an open set contained in $A \cup B$, therefore $x \in (A \cup B)^o$. A similar argument holds for the case of $x \in O_B \subseteq B$.

□

THEOREM 2.2.11 (8.37i). *Let $A, B \subseteq R^n$. Then*

$$(A \cap B)^\circ = A^\circ \cap B^\circ$$

PROOF. Suppose $\in A^o \cap B^o$. Then $x \in O_A$ for some $O_A \subseteq A$ and $x \in O_B$ for some $O_B \subseteq B$. Therefore, $x \in O_A \cap O_B$. By the properties of a topology, finite intersections of open sets are open, so $O_A \cap O_B$ is open and in fact is an open set contained in $A \cap B$. So, by definition, $x \in (A \cap B)^o$.
Now, suppose $x \in (A \cap B)^o$. Then $x \in O_{A\cap B} \subseteq A \cap B$ by deinition. But $O_{A\cap B} \subseteq A \cap B \subseteq A$, so $x \in O_{A\cap B} \subseteq A$ implies that $x \in A^o$. A similar argument shows $x \in B^o$. So $x \in A^o$ and $x \in B^o$ implies that $x \in A^o \cap B^o$. □

THEOREM 2.2.12 (8.37ii). *Let* $A, B \subseteq R^n$. *Then*

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

THEOREM 2.2.13 (8.37ii). *Let* $A, B \subseteq R^n$. *Then*

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

PROOF. Supposed $x \in \overline{A \cap B}$.
Then x belongs to *every* closed set that contains $A \cap B$. But $A \cap B \subseteq A$, so every closed set that contains A also contains $A \cap B$. Therefore x is in every closed set that contains A. Further concluding, $x \in \overline{A}$.
By similar logic, x belongs to *every* closed set that contains $A \cap B$. But $A \cap B \subseteq B$, so every closed set that contains B also contains $A \cap B$. Therefore x is in every closed set that contains B. Further concluding, $x \in \overline{B}$.
Therefore $x \in \overline{A} \cap \overline{B}$, proving $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. □

THEOREM 2.2.14 (8.37iii). *Let* $A, B \subseteq R^n$. *Then*

$$\partial(A \cup B) \subseteq \partial A \cup \partial B$$

THEOREM 2.2.15 (8.37iii). *Let* $A, B \subseteq R^n$. *Then*

$$\partial(A \cap B) \subseteq \partial A \cap \partial B$$

CHAPTER 3

Convergence in $\mathbb{R}^n$

3.1. Limits of Sequences

DEFINITION 3.1.1 (convergent sequence). *Let $\{x_k\}$ be a sequence of points in $\mathbb{R}^n$. $\{x_n\}$ is said to converge to some point $a \in \mathbb{R}^n$, called the limit of x_k, if and only if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that*

$$k \geq N \quad \textit{implies} \quad \|x_k - a\| < \epsilon$$

In this case, we write $x_k \to a$ as $k \to \infty$ or $a = \lim_{k\to\infty} x_k$.

DEFINITION 3.1.2 (bounded sequence). *Let $\{x_k\}$ be a sequence of points in $\mathbb{R}^n$. $\{x_n\}$ is said to be bounded if and only if there is an $M > 0$ such that*

$$\|x_k\| \leq M \quad \textit{for all} \quad k \in \mathbb{N}$$

DEFINITION 3.1.3 (Cauchy sequence). *Let $\{x_k\}$ be a sequence of points in $\mathbb{R}^n$. $\{x_n\}$ is said to be Cauchy if and only if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that*

$$k, m \geq N \quad \textit{imply} \quad \|x_k - x_m\| < \epsilon$$

DEFINITION 3.1.4 (separable set). *$E \subset \mathbb{R}^n$ is said to be separable if, there is an at most countable subset $Z \subseteq E$ such that for every $a \in E$, there is a sequence $\{x_k\} \in Z$ that converges to a.*

THEOREM 3.1.1. *(9.2) Let $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and suppose*

$$\left\{x_k = \left(x_k^{(1)}, x_k^{(2)}, \dots, x_k^{(n)}\right)\right\} \quad k \in \mathbb{N}$$

be a sequence in $\mathbb{R}^n$. Then

$$x_k \to a \quad \textit{as} \quad k \to \infty$$

if and only if, for each $j \in \{1, 2, \dots, n\}$, the component sequence

$$x_k^{(j)} \to a_j \quad \textit{as} \quad k \to \infty$$

THEOREM 3.1.2. *(9.3) Let*

$$\mathbb{Q}^n = \{x \in \mathbb{R}^n : x_j \in \mathbb{Q} \quad \text{for} \quad j = 1, 2, \ldots, n\}$$

For each $a \in \mathbb{R}^n$, there is a sequence $x_k \in \mathbb{Q}^n$ such that $x_k \to a$ as $k \to \infty$.

PROOF. Let $a \in \mathbb{R}^n = (a_1, a_2, \ldots, a_n)$, $a_i \in \mathbb{R}$
There is a sequence $q_k^{(i)}$ in $\mathbb{Q}$ that converges to a_i for $1 \leq i \leq n$.
By Theorem 3.1.1, each component sequence $q_k^{(i)} \to a_i$ as $k \to \infty$, so the sequence $q_k \to a$ in $\mathbb{R}^n$. □

THEOREM 3.1.3. *$\mathbb{R}^n$ is separable.*

THEOREM 3.1.4. *(9.4i) A sequence in $\mathbb{R}^n$ can have at most one limit.*

THEOREM 3.1.5. *(9.4ii) If $\{x_k\}$ is sequence in $\mathbb{R}^n$ that converges to a as $k \to \infty$, then every subsequence $\{x_{k_j}\}$ also converges to a as $j \to \infty$.*

PROOF. Let $\epsilon > 0$ be given. By hypothesis, $x_k \to L_x$ and $y_k \to L_y$, so $\exists N \in \mathbb{N}$ such that $||x_k - L_x|| < \frac{\epsilon}{2}$ and $||y_k - L_y|| < \frac{\epsilon}{2}$ when $k \geq N$. But $||(x_k + y_k) - (L_x + L_y)|| = ||(x_k - L_x) + (y_k - L_y)|| \leq ||x_k - L_x|| - ||y_k - L_y|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ when $k \geq N$. □

THEOREM 3.1.6. *(9.4iii) Every convergent sequence in $\mathbb{R}^n$ is bounded. The converse of this statement is false.*

PROOF. If $x_k \to a$, then there exists an $N \in \mathbb{N}$ such that $||x_k - a|| < 1$ for all $k \geq N$.
(Note we are theoretically letting $\epsilon = 1$).
Now consider $\delta_i = ||x - a||$ for $i < i < N - 1$.

Let $m = \max(\delta_i)$.
Then $\mathrm{d}(a, x_i) \leq m + 1$ for all $i \in \mathbb{N}$. Thus $||a - x_i|| = \delta_i \leq m + 1$.
But, $||x_i - a|| \geq ||x|| - ||a||$.
So, $||x|| - ||a|| \leq m + 1 \quad \Rightarrow \quad ||x_i|| \leq ||a|| + m + 1$. □

THEOREM 3.1.7. *(9.4iv) Every convergent sequence in $\mathbb{R}^n$ is Cauchy.*

PROOF. Suppose x_n is a convergent sequence in $\mathbb{R}^n$, and let $\epsilon > 0$ be given. By hypothesis, $x_N \to L$ so $\exists N \in \mathbb{N}$ such that $\|x_k - L\| < \frac{\epsilon}{2}$ when $k \geq N$.

$$\|x_k - L\| < \epsilon$$

$$\|x_k - x_N\| < \epsilon$$

$$\|x_k - L + L - x_N\| \leq \|x_k - L\| + \|L - x_N\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

THEOREM 3.1.8. *(9.4va) If $\{x_k\}$ and $\{y_k\}$ are convergent sequence in $\mathbb{R}^n$, then*

$$\lim_{k\to\infty} (x_k + y_k) \ = \ \lim_{k\to\infty} x_k + \lim_{k\to\infty} y_k$$

PROOF. Let $\epsilon > 0$ be given. By hypothesis, $\{x_k\} \to L_x$ and $\{y_k\} \to L_y$, so $\exists N \in \mathbb{N}$ such that $||x_k - L_x|| \leq \frac{\epsilon}{2}$ and $\|y_k - L_y\| \leq \frac{\epsilon}{2}$ when $k \geq N$. But then for $k \geq N$,

$$\|(x_k+y_k)-(L_x+L_y)\| = \|(x_k-L_x)+(y_k-L_y)\| \leq ||x_k-L_x||+||y_k-L_y|| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$$

when $k \geq N$. □

THEOREM 3.1.9. *(9.4vb) If $\{x_k\}$ is a convergent sequence in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then*

$$\lim_{k\to\infty} (\alpha x_k) \ = \ \alpha \lim_{k\to\infty} x_k$$

PROOF. Let $\epsilon > 0$ be given. We need to find $N \in \mathbb{N}$ such that

$$\|\alpha x_k - \alpha L_x\| < \epsilon \text{ when } k \geq N.$$

By hypothesis, $x_k \to L_x$, so there exists $N \in \mathbb{N}$ such that

$$\|x_k - L_x\| < \frac{\epsilon}{|\alpha|} \text{ when } k > N$$

Then for $k > N$,

$$\|\alpha x_k - \alpha L_x\| = |\alpha| \|x_k - L_x\| < |\alpha| \frac{\epsilon}{|\alpha|} = \epsilon$$

□

THEOREM 3.1.10. *(9.4vc) If $\{x_k\}$ and $\{y_k\}$ are convergent sequence in $\mathbb{R}^n$, then*

$$\lim_{k\to\infty} (x_k \cdot y_k) \;=\; (\lim_{k\to\infty} x_k) \cdot (\lim_{k\to\infty} y_k)$$

THEOREM 3.1.11. *If $\{x_k\}$ is convergent sequence in $\mathbb{R}^n$, then*

$$\lim_{k\to\infty} \|x_k\| \;=\; \| \lim_{k\to\infty} x_k\|$$

PROOF. Using the triangle inequality: $||x - y|| \geq ||x|| - ||y||$, we say:

$$||x_n - L|| \geq ||x_n|| - ||L||$$

$$\Rightarrow \; ||x_n - L|| + ||L|| \geq ||x_n|| \quad \forall \; n$$

Taking the limit:

$$\lim_{n\to\infty} ||x_n - L|| + \lim_{n\to\infty} ||L|| \geq \lim_{n\to\infty} ||x_n||$$

But, $\lim_{n\to\infty} ||x_n - L|| \to 0$, thus:

$$||L|| \geq \lim_{n\to\infty} ||x_n||$$

Reversing, $||L - x_n|| \geq ||L|| - ||x_n||$

$$\Rightarrow \; ||L - x_n|| - ||L|| \geq -\, ||x_n||$$

$$\Rightarrow \; ||L|| - ||L - x_n|| \leq ||x_n||$$

Taking the limit:

$$\lim_{n\to\infty} ||L|| - \lim_{n\to\infty} ||L - x_n|| \leq \lim_{n\to\infty} ||x_n||$$

Once again, $\lim_{n\to\infty} ||L - x_n|| \to 0$, so:

$$||L|| \leq \lim_{n\to\infty} ||x_n||$$

Thus:

$$||L|| \leq \lim_{n\to\infty} ||x_n|| \leq ||L||$$

Concluding:

$$\lim_{n\to\infty} ||x_n|| = ||L||$$

□

THEOREM 3.1.12 (Bolzano-Weierstrass). *(9.6) Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

PROOF. By hypothesis, $\{x_k\}$ is bounded, so there exists an $M > 0$ such that $||x_k|| \leq M$ for all $k \in \mathbb{N}$.

By Theorem 2.1.8, $|x_{k_j}| \leq max(|x_{k_1}|, |x_{k_2}|, ..., |x_{k_n}|) = ||x||_\infty \leq ||x||$ for all $k \in \mathbb{N}$. So each component sequence $\{x_{k_j}\}$ with $k = 1, 2, 3, ...$ and $1 \leq j \leq n$, is bounded. Starting with $\{x_{k_1}\}$, the sequence of first components, by the Bolzano-Weierstrass Theorem in $\mathbb{R}$, $\{x_{k_1}\}$ has a convergent subsequence, $\{x_{k_{1_l}}\}$. Starting with each of the $\{x_k\}$, elements whose first component x_{k_1} is in the convergent subsequence of first components, choose a subsequence so that the sequence of second elements is convergent. Continue in this fashion, constructing subsequences of $\{x_k\}$ for which the first, second, and third components form convergent sequences in $\mathbb{R}$, then the first, second, third, and fourth, and so on until each component forms a convergent sequence. By an earlier theorem, this means the vector subsequence converges.

□

THEOREM 3.1.13. *(9.6) A sequence $\{x_k\}$ in $\mathbb{R}^n$ is convergent if and only if it is Cauchy.*

PROOF. Suppose $\{x_n\}$ is Cauchy.
Given $\epsilon = 1$, let us choose $N \in \mathbb{N}$ such that

$$||x_n - x_m|| < 1 \text{ for all } n, m > N$$

By the Triangle Inequality,

$$||x_n|| - ||x_m|| \leq ||x_n - x_m|| < 1$$

$$\Rightarrow \ ||x_n|| \leq 1 + ||x_m||$$

Therefore, the sequence $\{x_n\}$ is bounded by

$$\max\{||x_1||, ||x_2||, \ldots, ||x_N - 1||, 1 + ||x_m||\}.$$

By the Bolzano-Weierstrass Theorem, we conclude $\{x_n\}$ has a convergent subsequence. So:

$$\{x_{n_k}\} \ \Rightarrow \ \exists\ K \in \mathbb{N} \text{ such that } ||x_{n_k} - L|| < \frac{\epsilon}{2} \text{ when } k \geq K$$

$$||x_m - x_{n_k}|| < \frac{\epsilon}{2} \text{ for } m, n_k > N$$

Thus:

$$||x_m - L|| \leq ||x_m - x_{n_k}|| + ||x_{n_k} - L|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ when } k > K \text{ and } m > N$$

Therefore, we conclude $x_n \to L$.

The converse has been established in Theorem 3.1.7. □

THEOREM 3.1.14. *(9.7) Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. Then $x_k \to a$ if and only if for every open set V that contains a, there is an $N \in \mathbb{N}$ such that*

$$k \geq N \quad \textit{implies} \quad x_k \in V$$

THEOREM 3.1.15. *(9.8) $E \subseteq \mathbb{R}^n$. Then E is closed if and only if E contains all of its limit points, that is, if and only if*

$$x_k \in E \quad \textit{and} \quad x_k \to a \quad \textit{implies} \quad a \in E$$

PROOF. Let $F \subseteq \mathbb{R}^n$ be a closed set and L a limit point of F. Then by definition, every open ball $B_\epsilon(L)$ contains points of F other than L. This implies that for every $\epsilon > 0$, $B_\epsilon(L) \nsubseteq F^x$. By hypothesis, F is closed, so F^c is open, and by definition if $L \in F^c$, $\exists > 0$ such that $B_\epsilon(L) \subseteq F^x$, contradicting that L is a limit point of F, therefore, $L \in F$.

If F contains its limit points, then F is closed. [L is a limit point of F] $\Rightarrow (L \in F)] \Rightarrow F$ is closed

$$P \Rightarrow Q \equiv \ P \vee Q$$

$$(P \Rightarrow Q) \equiv P \wedge \ Q$$

F^c is not open $\Rightarrow$ (L is a limit point of F and $L \notin F$). Exists for some $L \in F^c$, for which every neighborhood of L contains a point of $F^{c^c} = F$. For some $L \in F^c$, L is a limit point of F. $\square$

3.2. The Heine-Borel Theorem

DEFINITION 3.2.1 (open covering). *An open covering of* $E \subseteq \mathbb{R}^n$ *is a collection of sets* $\{V_\alpha\}_{\alpha \in A}$ *such that each* V_α *is open and*

$$E \subseteq \bigcup_{\alpha \in A} V_\alpha$$

DEFINITION 3.2.2 (finite subcovering). *If* $\{V_\alpha\}_{\alpha \in A}$ *is an open covering of* $E \subseteq \mathbb{R}^n$*, a finite subcovering is a finite collection*

$$A_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \quad \textit{such that} \quad E \subseteq \bigcup_{i=1}^{n} V_{\alpha_j}$$

DEFINITION 3.2.3 (compact set). *A set* $E \subseteq \mathbb{R}$ *is compact if and only if every open covering of* E *has a finite subcovering.*

LEMMA 3.2.1 (Borel covering lemma). *(9.9) Let* E *be a closed, bounded subset of* $\mathbb{R}^n$*. If* $r : E \to (0, \infty)$ *is an arbitrary function, then there exist finitely many points* $y_1, \ldots, y_n$ *such that*

$$E \subseteq \bigcup_{j=1}^{n} B_{r(y_j)}(y_j)$$

THEOREM 3.2.1 (Heine-Borel). *(9.11)* $E \subseteq \mathbb{R}^n$ *is compact if and only if it is closed and bounded.*

PROOF. Suppose $E \subseteq \mathbb{R}^n$ is closed and bounded, and O_α, $\alpha \in A$, is an open cover of E. By hypothesis, O_α, $\alpha \in A$, is an open cover of E, so every element of E belongs to $\bigcup_{\alpha \in A} O_\alpha$. Since $\bigcup_{\alpha \in A} O_\alpha$ is open set itself, there is an $\epsilon_y > 0$ for every $y \in E$ such that

$$B_{\epsilon_y} \subseteq \bigcup_{\alpha \in A} O_\alpha \text{ and } E \subseteq \bigcup_{y \in E} B_{\epsilon_y}(y)$$

Since $r : y \to \epsilon_y$ is a function from E to $(0, \infty)$, By the Borel covering lemma, there exist a finite collection of the $B_{\epsilon_y}(y)$ such that:

$$E \subseteq \bigcup_{i=1}^{n} B_{\epsilon_y}(y)$$

Since each $B_{\epsilon_y}(y) \subseteq O_\alpha$ for some $\alpha \in A$ there is a finite collection of O_α's that

$$\bigcup_{i=1}^{n} O_{\alpha_i} \supseteq \bigcup_{i=1}^{n} B_{\epsilon_y}(y_i) \supseteq E$$

. Since O_α, $\alpha \in A$, was arbitrary, every open cover of E has a finite subcover, and by definition, E is compact. □

3.3. Limits of Functions

DEFINITION 3.3.1 (function convergence). *(9.14) Let* $n, m \in \mathbb{N}$ *and* $a \in \mathbb{R}^n$*, and let* V *be an open set that contains* a*. If* f *is a function*

$$f : V \setminus \{a\} \to \mathbb{R}^m$$

we say that $f(x)$ *converges to* L *as* x *approaches* a *if and only if, for every* $\epsilon > 0$ *there is a* $\delta > 0$ *(which in general depends on* ϵ*,* f*,* V*, and* a*) such that*

$$0 < \|x - a\| < \delta \quad \textit{implies} \quad \|f(x) - L\| < \epsilon$$

When this is the case, we write

$$f(x) \to L \quad \textit{as} \quad x \to a \quad \textit{or} \quad L = \lim_{x \to a} f(x)$$

and call L *the limit of* f *as* x *approaches* a*.*

DEFINITION 3.3.2 (iterated limits). *Let* V *be an open subset of* $\mathbb{R}^2$ *and* $(a, b) \in V$*. The iterated limits of* f *at* (a, b) *are defined to be:*

$$\lim_{x \to a} \lim_{y \to b} f(x, y) \;=\; lim_{x \to a} \left(\lim_{y \to b} f(x, y) \right)$$

and

$$\lim_{y \to b} \lim_{x \to a} f(x, y) \;=\; lim_{y \to b} \left(\lim_{x \to a} f(x, y) \right)$$

THEOREM 3.3.1. *(9.15i) Suppose* $a \in \mathbb{R}^n$*,* V *is an open set that contains* a*, and* $f, g : V \setminus \{a\} \to \mathbb{R}^m$*. If*

$$f(x) = g(x) \quad \textit{for all} \quad x \in V \setminus \{a\}, \quad \textit{and} \quad \lim_{x \to a} f(x) \quad \textit{exists}$$

then

$$\lim_{x \to a} g(x) \quad \textit{exists and} \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

THEOREM 3.3.2. *(9.15ii) [sequential characterization of limits] Suppose $a \in \mathbb{R}^n$, V is an open set that contains a, and $f : V \setminus \{a\} \to \mathbb{R}^m$. Then*

$$\lim_{x\to a} f(x) = L \quad \textit{if and only if} \quad f(x_k) \to L \quad \textit{as} \quad k \to \infty$$

for every sequence $x_k \in V \setminus \{a\}$ that converges to a as $k \to \infty$.

PROOF. First we can assume: $f(x) \to L$ as $x \to a$ for any sequence $\{x_n\}$ with $x_n \to a$ as $n \to \infty$.

$\Rightarrow$ **Proof**
For any $\epsilon > 0$, $\exists$ N such that

$$|f(x_n) - L| < \epsilon \text{ when } n > N$$

By our given information, we know $\exists$ $N \in \mathbb{N}$ such that

$$|x_n - a| < \delta \text{ when } n \geq N$$

Then for $n \geq N, |x_n - a| < \delta$.
By hypothesis, $x_n \in V \setminus \{a\}$, so $x_n \neq a$, and $0 < |x_n - a| < \delta$.
By hypothesis, $f(x) \to L$ as $x \to a$, so $f(x_n) \to L$ as $n \to \infty$ by definition.
Since this is true for each $n \geq N$, we have:

$$\lim_{n\to\infty} f(x_n) \to L$$

□

THEOREM 3.3.3. *(9.15iiia) If $f(x)$ and $g(x$ have limits as $x \to a$, then*

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

PROOF. $\forall\epsilon > 0$, $\exists\delta > 0$ such that:

$$||(f+g)(x) - (L+M)|| = ||(f(x)+g(x)-L-M|| = ||(f(x)-L)+(g(x)-M)|| < \epsilon \quad \text{when} \quad ||x-a||$$

By triangle inequality,

$$||(f(x) - L) + (g(x) - M)|| \leq ||f(x) - L|| + ||g(x) - M||$$

So, choose δ such that

$$||f(x) - L|| < \frac{\epsilon}{2} \quad \text{and} \quad ||g(x) - M|| \leq \frac{\epsilon}{2}$$

Then, for $||x - a|| < \delta$,

$$||(f+g)(x) - (L+M)|| \leq ||f(x) - L|| + ||g(x) - M|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

THEOREM 3.3.4. *(9.15iiib) If $f(x)$ has a limit as $x \to a$, then*

$$\lim_{x\to a}(\alpha f)(x) = \alpha \lim_{x\to a} f(x)$$

THEOREM 3.3.5. *(9.15iiic) If $f(x)$ and $g(x$ have limits as $x \to a$, then*

$$\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

THEOREM 3.3.6. *(9.15iiid) If $f(x)$ has a limit as $x \to a$, then*

$$\|\lim_{x\to a}(f)(x)\| = \lim_{x\to a}\|f(x)\|$$

THEOREM 3.3.7. *(9.15iv) [squeeze theorem for functions] Suppose $f, g, h : V \setminus \{a\} \to \mathbb{R}$ and*

$$g(x) \leq h(x) \leq f(x) \quad \textit{for all} \quad x \in V \setminus \{a\}$$

If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$$

then the limit of h as x approaches a also exists, and

$$\lim_{x\to a} h(x) = L$$

THEOREM 3.3.8. *(9.15v) Suppose $U \subset \mathbb{R}^m$ is open, $L \in U$, and*

$$h : U \setminus \{L\} \to \mathbb{R}^p \quad \textit{for some} \quad p \in \mathbb{N}$$

If

$$\lim_{x \to a} g(x) = L \quad \textit{and} \quad \lim_{y \to L} h(y) = M$$

then

$$\lim_{x \to a} (h \circ g)(x) = M$$

THEOREM 3.3.9. *(9.16) Let $a \in \mathbb{R}^n$, let V be an open set that contains a, and suppose*

$$f = (f_1, \ldots, f_m) : V \setminus \{a\} \to \mathbb{R}^m$$

then

$$\lim_{x \to a} f(x) = L = (L_1, \ldots, L_m)$$

exists in $\mathbb{R}^m$ if and only if

$$\lim_{x \to a} f_j(x) = L_j$$

exists for $j = 1, \ldots, m$

CHAPTER 4

Metric Spaces

4.1. Introduction

DEFINITION 4.1.1 (metric space). *A metric space is pair (X, ρ) consisting of a set X together with a function $\rho : X \times X \to \mathbb{R}$ called the metric of X which satisfies the following properties for all $x, y, z \in X$:*

positive definite	$\rho(x, y) \geq 0$ *with* $\rho(x, y) = 0 \Leftrightarrow x = y$
symmetric	$\rho(x, y) = \rho(y, x)$
triangle inequality	$\rho(x, y) leq \rho(x, z) + \rho(z, y)$

(Note: by definition, $\rho(x, y)$ is finite for all $x, y \in X$.

DEFINITION 4.1.2 (open ball). *The open ball in (X, ρ) with center a and radius r is the set*

$$B_r(a) = \{x \in X : \rho(x, a) < r$$

DEFINITION 4.1.3 (closed ball). *The closed ball in (X, ρ) with center a and radius r is the set*

$$B_r(a) = \{x \in X : \rho(x, a) \leq r$$

DEFINITION 4.1.4 (open set). *A set $V \subseteq X$ is said to be open if and only if for every $x \in V$, there is an $\epsilon > 0$ such that*

$$B_\epsilon(x) \subseteq V$$

DEFINITION 4.1.5 (closed set). *A set $E \subseteq X$ is said to be closed if and only if*

$$E^c = X \setminus E \quad \textit{is open}$$

DEFINITION 4.1.6 (convergent sequence). *Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ converges (in X) if there is a point $a \in X$ called the limit of x_n such that for every epsilon > 0, there is an $N \in \mathbb{N}$ such that*

$$\rho(x_n, a) < \textit{epsilon} \quad \textit{whenever} \quad n \geq N$$

DEFINITION 4.1.7 (Cauchy sequence). *Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if for every epsilon > 0, there is an $N \in \mathbb{N}$ such that*

$$\rho(x_n, x_m) < \textit{epsilon} \quad \textit{whenever} \quad n, m \geq N$$

DEFINITION 4.1.8 (bounded sequence). *Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is bounded if there is an $M > 0$ and a point $b \in X$ such that*

$$\rho(x_n, b) \leq M \quad \textit{for all} \quad n \in \mathbb{N}$$

DEFINITION 4.1.9 (complete metric space). *A metric space (X, ρ) is said to be complete if every Cauchy sequence in X converges to some point in X.*

THEOREM 4.1.1 (Example 10.2). *Every Euclidean space $\mathbb{R}^n$ is a metric space $(\mathbb{R}^n, \rho)$ where $\rho(x, y) = \|x - y\|$ is called the "usual metric on $\mathbb{R}^n$.*

THEOREM 4.1.2 (Example 10.3). *$\mathbb{R}$ is a metric space $(\mathbb{R}, \sigma)$ where*

$$\sigma(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

σ is called the discrete metric.

PROOF. Need to show: σ is a metric.

1) Let $x, y \in \mathbb{R}$. By defintion,

$$\sigma(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

So, $\sigma(x, y) \geq 0$ for all $x, y \in \mathbb{R}$, and
$\sigma(x, y) = 0$ iff $x = y$.

2) $\sigma(x, y) = \sigma(y, x)$
Case 1. If $x = y$, $\sigma(x, y) = 0 = \sigma(y, x)$

Case 2. If $x \neq y$, $\sigma(x, y) = 1 = \sigma(y, x,)$

3) For $x, y, z \in \mathbb{R}$, $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$

Case 1. $x = y, x \neq z$, $\sigma(x, y) = 0 \leq \sigma(x, z) + \sigma(y, z) = 1 + 1$

Case 2. $x = y = z$, $\sigma(x, y) = 0 = \sigma(x, z) + \sigma(y, z) = 0 + 0$

Case 3. $x \neq y$, $\sigma(x, y) = 1$ Either $x = z$ or $x \neq z$. If $x = z$, and $x \neq y, y \neq z$, so $\sigma(x, z) + \sigma(z, y) = 0 + 1$, $1 \leq 1$.

If $x \neq z$ and $x \neq y, \sigma(x, y) = 1 \leq \sigma(x, z) + \sigma(y, z) = \begin{cases} 1 & if \quad y = z \\ 2 & if \quad y \neq z \end{cases}$

□

THEOREM 4.1.3 (Example 10.4). *If (X, ρ) is a metric space and $E \subseteq X$, then (E, ρ) is a metric space.*

THEOREM 4.1.4 (Example 10.5). *$(\mathbb{Q}, \rho)$ is a metric space with $\rho(x, y) = |x - y|$.*

PROOF. Let $x, y, z \in \mathbb{Q}$. We need to show $|x - y|$ is a metric.

1) $|x - y| \geq 0$ by the definition of absolute value. $|x - y| = 0$ only if $x = y$, again by property of absolute value:

$$|x - y| = \begin{cases} x - y & \text{if} \quad x - y \geq 0 \\ y - x & \text{if} \quad x - y < 0 \end{cases}$$

This can only be zero if $x - y = 0 \Rightarrow x = y$.

2) $|x - y| = |y - x|$. By definition of absolute value, $|a| = |-a|$, so $|x - y| = |y - x|$

3) By the triangle inequality for real numbers, $|x - y| \leq |x - z| + |z - y|$

□

THEOREM 4.1.5 (Example 10.6). *Let $\mathcal{C}[a, b]$ be the set of continuous real-valued function on $[a, b]$, that is, the collection of all functions $f : [a, b] \to \mathbb{R}$ continuously and let*

$$\|f\| = \sup_{x \in [a,b]} |f(x)|$$

Then $\mathcal{C}[a, b], \rho)$ *is a metric space with* $\rho(f, g) = \|f - g\|$ *for* $f, g \in \mathcal{C}[a, b]$.

PROOF. Note: Definition of a metric: $(X, \rho) : X \times X \to \mathbb{R}$ such that $\forall x, y \in X$:

$$1)\rho(x, y) \geq 0 \text{ with } \rho(x, y) = 0 \leftrightarrow x = y$$

$$2)\rho(x, y) = \rho(y, x)$$

$$3)\rho(x, y) \leq \rho(x, z) + \rho(z, y)$$

Let $X = C[a, b]$ with $\rho(f, g) = ||f - g||$ and $||f|| = \sup |f(x)|$.
2) Since $|f - g| = |g - f| \forall x \in [a, b], \sup |f - g| = \sup |g - f|$. By definition, this implies $\rho(f, g) = \rho(g, f)$.
1) $|f - g| \geq 0 \forall x \in [a, b]$. If $f = g$, $|f - g| = |0| = 0 \forall x \in [a, b]$, so $\sup_{x \in [a,b]} |f - g| = 0$. Suppose $\rho(f, g) = 0$. This implies $\sup |f - g| = 0$. By definition of absolute values, $0 \leq \sup |f - g| = 0.0 \leq |f - g| = 0$, implying $f = g \forall x \in [a, b]$.
3) $\sup |f - g| \leq \sup |f - h| + \sup |h - g| \forall x \in [a, b]$. We know that $|f - g| = |f - h + h - g| \leq |f - h| + |h - g| \forall x \in [a, b]$. This implies $\sup |f - g| \leq \sup(|f - h| + |h - g|) \leq \sup |f - h| + \sup |h - g|$. Therefore $\rho(f, g) \leq \rho(f, h) + \rho(h, g)$.

□

THEOREM 4.1.6 (Example 10.9a). *Every open ball in* (X, ρ) *is open.*

THEOREM 4.1.7 (Example 10.9b). *Every closed ball in* (X, ρ) *is closed.*

THEOREM 4.1.8 (Example 10.10). *Singleton sets (sets consisting of a single element* $a \in X$*) are closed.*

THEOREM 4.1.9 (Remark 10.11). *In an arbitrary metric space* $(\mathbb{R}, \rho)$, X *and* $\emptyset$ *are both open and closed.*

THEOREM 4.1.10 (Example 10.12). *Every subset of the discrete space $(\mathbb{R}, \sigma)$ is both open and closed.*

THEOREM 4.1.11 (Theorem 10.14i). *A sequence in a metric space can have at most one limit.*

THEOREM 4.1.12 (Theorem 10.14ii). *If $x_n \in X$ converges to a, every subsequence x_{n_k} also converges to a.*

THEOREM 4.1.13 (Theorem 10.14iii). *Every convergent sequence in a metric space is bounded.*

THEOREM 4.1.14 (Theorem 10.14iv). *Every convergent sequence in a metric space is Cauchy.*

THEOREM 4.1.15 (Theorem 10.15). *A sequence $x_n \in X$ converges to a if and only if for every open set V that contains a, there is an $N \in \mathbb{N}$ such that*

$$x_n \in V \quad \text{whenever} \quad n \geq N$$

PROOF. $\Rightarrow$ By hypothesis, $x_n \to a$. Let V be an open set that contains a. By definition of an open set, $\exists \epsilon > 0$ such that $B_\epsilon(a) \subseteq V$. Since $x_n \to a$ as $n \to \infty$, there is an $N \in \mathbb{N}$ such that $\rho(x_n, a) < \epsilon$ when $n \geq N$. This implies $x_n \in B_\epsilon(a)$ when $n \geq N$.

$\Leftarrow$ Let $\epsilon > 0$ be given. Let V be an open set with $a \in V$. By definition, $\exists \delta > 0$ such that $B_\delta(a) \subseteq V$. Then $B_\delta(a)$ is an open set that contains a, so there is an $N \in \mathbb{N}$ such that for $n \geq N \implies x_n \in B_\delta(a)$. Likewise, $B_{\frac{\delta}{2}}(a)$ is an open set that contains a, so there is an $N \in \mathbb{N}$ such that $n \geq N \implies x_n \in B_{\frac{\delta}{2}}(a)$ when $n \to \infty$. Continuing in this fashion to $x_n \in B_{\frac{\delta}{2}^k}(a)$ when $n \geq N_k$ with $k \geq \log_2 \frac{\delta}{\epsilon}$, we have $\frac{\delta}{\epsilon}^k < \epsilon$. So, $x_n \in B_{\frac{\delta}{2}^k}(a)$ when $n \geq N_k$, implying $\rho(x_n, a) < \frac{\delta}{2}^k < \epsilon$.

□

THEOREM 4.1.16 (Theorem 10.16). *A subset E of the metric space (X, ρ) is closed if and only if the limit of every convergent sequence in E belongs to E.*

THEOREM 4.1.17 (Remark 10.17). *The discrete metric space* $(\mathbb{R}, \sigma)$ *contains bounded sequences with no convergent subsequence.*

PROOF.

$$(\mathbb{R}, \sigma)$$

$$\sigma(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

In S, there exist bounded sequences with no convergent subsequence. Let $x \in \mathbb{R}$. For any $y \in \mathbb{R}$, with $y \neq x$, $\sigma(x, y) = 1$. Therefore, every sequence is bounded because $\sigma(x_n, x) \leq 1$. Let $\{x_n\} = \{1, 2, 3, 4, ...\} = \{\mathbb{N}\}$ for any $n \in \mathbb{N}$, if $\epsilon = \frac{1}{2}$, there does not exist any point a and integer N with $\sigma(x_n, a) < \frac{1}{2}$ when $n \geq N$. Therefore, $\{x_n\}$ does not converge. The same argument holds for any subsequence $\{x_{n_k}\}$.

□

THEOREM 4.1.18 (Remark 10.18). *The metric space* $(\mathbb{Q}, \rho)$ *contains Cauchy sequences that do not converge.*

PROOF. By counterexample, the sequence $1, 1.4, 1.414, 1.4142, 1.41421, \ldots$ in $\mathbb{R}$ converges to $\sqrt{2}$. But, $\sqrt{2} \notin \mathbb{Q}$. So the limit of this sequence does not belong to $\mathbb{Q}$ and we say it does not converge. □

THEOREM 4.1.19 (Theorem 10.21). *A subset E of a complete metric space (X, ρ) is a complete metric space if and only if E is closed.*

4.2. Cluster Points and Limits

DEFINITION 4.2.1 (cluster point). *A point $a \in X$ is said to be a cluster point of X if and only if $B_\delta(a)$ contains infinitely many points (of X) for each $\delta > 0$.*

DEFINITION 4.2.2 (function limit). *Let a be a cluster point of X and $f : x \setminus \{a\} \to Y$. Then f is said to converge to L as x approaces a if and only if, for every $\epsilon > 0$, there is a $\delta > 0$ such that*

$$0 < rho(x, a) < \delta \quad \Rightarrow \quad \tau(f(x), L) < \epsilon$$

f is said to be continuous on E if it is continuous at every $x \in E$.

THEOREM 4.2.1 (10.26i). *Let a be a cluster point of X and $f, g : X \setminus \{a\} \to Y$. If $f(x) = g(x)$ for all $x \in X \setminus \{a\}$, and $f(x)$ has a limit as $x \to a$, then $g(x)$ also has a limit as $x \to a$ and*

$$\lim_{x \to a} f(x) \;=\; \lim_{x \to a} g(x)$$

4.3. Compactness

DEFINITION 4.3.1 (compactness). *A subset H of a metric space X is said to be compact if and only if every open covering of H has a finite subcover.*

DEFINITION 4.3.2 (separable). *A metric space X is said to be separable if and only if it contains a countable dense subset (i.e., iff there is a countable subset Z of X such that for every point $A \in X$ there is a sequence $x_k \in Z$ such that $x_k \to a$ as $k \to \infty$.*

THEOREM 4.3.1 (Remark 10.43). *The empty set and all finite subsets of a metric space are compact.*

PROOF. **Part 1** $\emptyset \subseteq X$, i.e., (X, ρ).
Let $O \subseteq \cup_{\alpha \in A} O_\alpha$ be any non-empty collection of open subsets of X. Pick any element O_α. Then, $\emptyset \subseteq O_\alpha$, so O_α is a finite open cover of $\emptyset$, with one element. Since we can do this for any open cover of $\emptyset$, the empty set, $\emptyset$, is compact.

Part 2 Finite Subsets
Let E be a finite subset of X, and $0 = \cup_{\alpha \in A} O_\alpha$ an open cover. That is, $E \subseteq \cup_{\alpha \in A} O_\alpha$. Let x_i for $i = 1, 2, 3, \ldots, N$ be the finite elements of E. Every x_i belongs to O, so every x_i belongs to at least one O_α. Let $x_1 \in O_{\alpha_1}, x_2 \in O_{\alpha_2}, \ldots, x_N \in O_{\alpha_N}$. Then $\cup_{i=1}^{N} O_{\alpha_i}$ is a finite subcover containing E. Since O was arbitrary, we can find such a subcover for any open cover.

□

THEOREM 4.3.2 (Remark 10.44). *In a metric space a compact set is always closed.*

THEOREM 4.3.3 (Remark 10.45). *A closed subset of a compact set is compact.*

Theorem 4.3.4 (10.46). *Let H be a subset of a metric space X. If H is compact, then H is closed and bounded.*

Theorem 4.3.5 (Remark 10.47). *The converse of the previous theorem is false.*

Theorem 4.3.6 (10.49 Lindelof). *Let E be a subset of a separable metric space X. If $\{V_\alpha\}_{\alpha\in A}$ is a collection of open sets and $E \subseteq \bigcup_{\alpha\in A} V_\alpha$ then there is a countable subset $\{\alpha_1, \alpha_2, \ldots\}$ of A such that*

$$E \subseteq \bigcup_{k=1}^{\infty} V_{\alpha_k}$$

Theorem 4.3.7 (10.50 Heine-Borel). *Let X be a separable metric space which satisfies the Bolzano-Weierstrass Property, and H a subset of a X. Then H is compact if and only if it is closed and bounded.*

4.4. Function Algebras and the Stone-Weierstrass Theorem

DEFINITION 4.4.1 (uniform continuity). *Let X be a metric space, E a nonempty subset of X, and $f : E \to Y$. Then f is said to be uniformly continuous on E if and only if given $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\rho(x, a) < \delta \quad \textit{and} \quad x, a \in E \quad \textit{imply} \quad \tau(f(x), f(a)) < \epsilon$$

DEFINITION 4.4.2 (algebra). *A subset A of $\mathcal{C}(X)$ is said to be a (real function) algebra in $\mathcal{C}(X)$ if and only if*

- $\emptyset \neq A \subseteq \mathcal{C}(X)$
- *If $f, g \in A$, then $f + g$ and fg belong to A*
- *If $f \in A$ and $c \in \mathbb{R}$, then $cf \in A$.*

DEFINITION 4.4.3 (uniformly closed). *$A \subseteq \mathcal{C}(X)$ is said to be (uniformly) closed if and only if for every sequence $f_n \in A$ satisfying*

$$\|f_n - f\| \to 0 \quad \textit{as} \quad n \to \infty \;\Rightarrow\; \lim f_n = f \in A$$

DEFINITION 4.4.4 (uniformly dense). *$A \subseteq \mathcal{C}(X)$ is said to be (uniformly) dense if and only if given $\epsilon > 0$ and $f \in \mathcal{C}(X)$, there is a function*

$$g \in A \quad \textit{such that} \quad \|g - f\| < \epsilon$$

DEFINITION 4.4.5 (separates points). *$A \subseteq \mathcal{C}(X)$ separates points if and only if, given $x, y \in X$ with $x \neq y$, there exists an $f \in A$ such that*

$$f(x) \neq f(y)$$

THEOREM 4.4.1 (10.52). *If E is a compact subset of X and $f : X \to Y$. Then f is uniformly continuous on E if and only if it is continuous on E.*

THEOREM 4.4.2 (10.58). *Suppose $f : X \to Y$. Then f is continuous if and only if $f^{-1}(V)$ is open in X for every open $V \subseteq Y$.*

THEOREM 4.4.3 (10.61). *If H is compact in X and $f : H \to Y$ is continuous on H, then $f(H)$ is compact in Y.*

THEOREM 4.4.4 (10.63 Extreme Value Theorem). *Let H be a nonempty, compact subset of a metric space X. If $f : H \to \mathbb{R}$ is continuous, then*

$$M = \sup\{f(x) : x \in H\} \quad \textit{and} \quad m = \int \{f(x) : x \in H\}$$

are finite real numbers and there exist points x_M and x_m in H such that

$$M = f(x_M) \quad \textit{and} \quad m = f(x_m)$$

THEOREM 4.4.5 (10.64). *If H is a compact subset of X and $f : H \to Y$ is $1-1$ and continuous, then f^{-1} is continuous on $f(H)$.*

THEOREM 4.4.6 (10.69 Stone-Weierstrass). *Suppose X is a compact metric space. If A is an algebra in $\mathcal{C}(X)$ that separates the points of X and contains the constant functions, then A is uniformly dense in $\mathcal{C}(X)$*