Random Vectors and Sums

Suppose X is a vector of two random variables representing a roll of two balanced dice,

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where X_1 and X_2 each take the values $1, 2, 3, 4, 5, 6$.

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In this case, we define a new random variable Y as the sum of X_1 and X_2:

$$Y = X_1 + X_2$$

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We often need to determine the expected value $E(Y)$ and variance $V(Y)$ for this new variable.

We will assume that we know the expected value μ of the random vector X

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix}$$

and its variance-covariance matrix V:

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

Random Vectors and Sums

The most succinct way to determine $E(Y)$ and $V(Y)$ is with matrix algebra. Think of the sum as the matrix product

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More generally, we can represent any **linear combination** of X_1 and X_2

$$[a_1,a_2]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = a\cdot X_1 + b\cdot X_2 = a_1X_1 + a_2X_2$$

Random Vectors and Sums

If we write the coefficients $[a_1, a_2]$ as a column vector a,

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

then the linear combination $Y = a_1X_1 + a_2X_2$ can be written in matrix form

$$Y = a'X = [a_1, a_2] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

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The symbol a' represents the **transpose** of a, which we obtain by writing the columns of a as rows:

$$\text{if} \quad a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{then} \quad a' = [a_1, a_2]$$

Random Vectors and Sums

The following important result gives us a way to compute $E(Y)$ and $V(Y)$: Suppose

$$Y = a'X \quad \text{with} \quad a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and

$$E(X) = \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

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Then

$$E(Y) = a'\mu \quad \text{and} \quad V(Y) = a'Va$$

Random Vectors and Sums

Example: Suppose X is a two-dimensional random vector:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad E(X) = \mu = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

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If we define a new random variable $Y = X_1 + X_2$, then

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then

$$E(Y) \; = \; a'\mu = [1, 1] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \cdot 1 + 1 \cdot 3 = 4$$

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and

$$v(Y) = a'Va = [1,1]\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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$$= [3,4]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7$$

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$$= [3,4]\begin{bmatrix} 1 \\ 1 \end{bmatrix} \;=\; 7$$

Although the matrix notation can be confusing at first, the formulas

$$E(a'X) \;=\; a'\mu \quad \text{and} \quad \text{Variance}(a'X) \;=\; a'Va$$

are much simpler than their non-matrix counterparts.

Random Vectors and Sums

Packages like MATLAB and GNU OCTAVE are designed to perform matrix computations and are optimized for it.

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While not primarily designed for this, R has some rudimentary facilities that will suffice for our purposes.

The preceding example would be programmed in R as:

```
mu=matrix(c(1,3),nrow=2)
V=matrix(c(2,1,1,3),nrow=2
a=matrix(c(1,1),nrow=2)
t(a)%*%mu
t(a)%*%V%*%a
```

Example 1

If X is a two-dimensional random vector and $Y = X_1 + X_2$, find $E(Y)$ and $V(Y)$ if

$$E(X) = \mu = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix}$$

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$$E(X) = \mu = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix}$$

Solution: In this case, $a = [1, 1]'$ and

$$E(Y) = a'\mu = [1, 1] \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 6$$

$$V(Y) = a'Va = [1, 1] \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5$$

Example 2

If X is a two-dimensional random vector and $Y = X_1 - X_2$, find $E(Y)$ and $V(Y)$ if

$$E(X) = \mu = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix}$$

Example 2

If X is a two-dimensional random vector and $Y = X_1 - X_2$, find $E(Y)$ and $V(Y)$ if

$$E(X) = \mu = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix}$$

Solution: In this case, $a = [1, -1]'$ and

$$E(Y) = a'\mu = [1, -1] \begin{bmatrix} 2 \\ 4 \end{bmatrix} = -2$$

$$V(Y) = a'Va = [1, -1] \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 13$$