Vectors of Discrete Random Variables

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The term **multivariate** is often used to describe the distributions we encounter working with random vectors.

Vectors of Discrete Random Variables

Multivariate discrete random variables (or random vectors) are a straightforward generalization of ordinary discrete random variables.

In place of a univariate probability mass function

$$p(x) \;=\; P(X = x)$$

we have a **joint** probability mass function, which in the 2-dimensional or bivarate case is:

$$p(x_1, x_2) \;=\; P(X_1 = x_1 \quad \text{and} \quad X_2 = x_2)$$

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In some cases, we will want to consider the component random variables $X1$ and X_2 as univariate or one-dimensinal random variables in their own right. These are called the **marginal** distributions of X_1 and X_2.

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Expected values are definied much the same way, except we have to sum over each dimension. In the bivariate case, the expected value of X_1, the first component, is defined by:

$$\mu_1 \;=\; E(X_1) \;=\; \sum_{R(X)} \sum_{R(Y)} x_1 \cdot p(x_1, x_2)$$

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The expected value of the second component X_2 is:

$$\mu_2 \;=\; E(X_2) \;=\; \sum_{R(X)} \sum_{R(Y)} x_2 \cdot p(x_1, x_2)$$

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Variances are defined similarly. In the bivariate case, the variance of X_1, the first component, is defined by:

$$\sigma_1^2 \ = \ V(X_1) \ = \ \sum_{R(X)} \sum_{R(Y)} (x_1 - \mu_1)^2 \cdot p(x_1, x_2)$$

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The variance of the second component X_2 is:

$$\sigma_2^2 = V(X_2) = \sum_{R(X)} \sum_{R(Y)} (x_2 - \mu_2)^2 \cdot p(x_1, x_2)$$

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In the multivariate case there is an additional construct called the **covariance** of X_1 and X_2 defined by

$$\sigma_{12} = Cov(X_1, X_2) = \sum_{R(X)} \sum_{R(Y)} (x_1 - \mu_1)(x_2 - \mu_2) \cdot p(x_1, x_2)$$

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The covariance is closely related to the **correlation coefficient** $\rho(X_1, X_2)$, which is defined as:

$$\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_1^2 \sigma_2^2}}$$

Vectors of Discrete Random Variables

It is usually convenient to use vector notation to describe a bivariate random variable and its expectation:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and

$$\mu = E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

Vectors of Discrete Random Variables

In the three dimensional case, the notation is:

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

and

$$\mu = E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ E(X_3) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

Vectors of Discrete Random Variables

Finally, the general case of n dimensions is:

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

and

$$\mu = E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

Vectors of Discrete Random Variables

In the bivariate case, $X = (X_1, X_2)$ and the generalization of the variance $V(X)$ is the **variance-covariance matrix** defined by

$$\Sigma = V(X) = \begin{bmatrix} V(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & V(X_2) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

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In the three dimensional case, $X = (X_1, X_2, X_3)$ and

$$\Sigma = V(X) = \begin{bmatrix} V(X_1) & Cov(X_1, X_2) & Cov(X_1, X_3) \\ Cov(X_1, X_2) & V(X_2) & Cov(X_2, X_3) \\ Cov(X_1, X_3) & Cov(X_2, x_3) & V(X_3) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{bmatrix}$$

Vectors of Discrete Random Variables

In the case of n dimensions, $X = (X_1, X_2, \ldots, X_n)$ and

$$\Sigma = V(X) = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & & \\ \sigma_{13} & \vdots & \sigma_3^2 & & \vdots \\ \vdots & & & \ddots & \\ \sigma_{1n} & & \cdots & & \sigma_n^2 \end{bmatrix}$$

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In n dimensions, there are

$$\frac{n(n+1)}{2} \quad \text{distinct variances and covariances}$$