Random Vectors

We previously introduced the idea of a **random vector** as a vector whose components are random variables:

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We suppose each of the random variables X_i has an expected value and variance, and each pair has a covariance:

$$E(X_i) = \mu_i \quad V(X_i) = \sigma_i^2 \quad Cov(X_i, X_j) = \sigma_{ij}$$

Random Vectors

We can think of the expected values as a vector of constants,

$$E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} = \mu$$

Random Vectors

It is usually convenient to represent the variances and covariances as a matrix,

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & & \sigma_n^2 \end{bmatrix}$$

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V is called the **variance-covariance matrix** of X.

Random Vectors

Among random vectors, the most important special case is one in which the components of X all have the same distribution, and are independent of each other.

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Among random vectors, the most important special case is one in which the components of X all have the same distribution, and are independent of each other.

In this case the X_i are said to be **independent, identically distributed** or IID.

If the components of X are IID, they all have the same expected value μ, so

$$E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} = \mu$$

Random Vectors

When the X_i are IID, the variance-covariance matrix simplifies dramatically.

Since independent random variables always have zero covariance, all off diagonal terms are zero.

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Since independent random variables always have zero covariance, all off diagonal terms are zero.

In addition, the variance of each X_i is the same, so for a vector of IID random variables each with variance $V(X_i) = \sigma^2$, we can write

$$V = \begin{bmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \ddots & \\ & & & \sigma^2 \end{bmatrix}$$

Random Vectors

In this case the matrix V has all off-diagonal terms equal to zero, and is said to be a **diagonal** matrix.

$$V = \begin{bmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \ddots & \\ & & & \sigma^2 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Random Vectors

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The diagonal matrix with all entries equal to one occurs frequently. It is denoted by I and is called the **identity matrix** because, for any $n \times n$ square matrix A,

$$AI = IA = A$$

Random Vectors

In summary, in the case of vector X of IID random variables each with expected value μ and variance σ^2,

$$E(X) = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} = \mu \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \ddots & \\ & & & \sigma^2 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \sigma^2 I$$

Linear Combinations of IID Random V

Linear combinations of IID random variables occur frequently because:

- Random samples are often considered to be IID random vectors
- The sample mean is a linear combination (all weights equal to $1/n$)

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- Random samples are often considered to be IID random vectors
- The sample mean is a linear combination (all weights equal to $1/n$)

In particular, the the expected value of the mean of an IID random vector X, which we can write as

$$E(\beta' X) \quad \text{with} \quad \beta = \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$$

Linear Combinations of IID Random V

From our pervious work with expected values of linear combinations, we know that

$$E(\beta' X) = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} = \sum_{i=1}^{n} \frac{\mu}{n} = \mu$$

Linear Combinations of IID Random V

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$$E(\beta' X) = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} = \sum_{i=1}^{n} \frac{\mu}{n} = \mu$$

The variance $Var(\beta' X)$ is $\beta' V \beta$, or

$$\begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \ddots & \\ & & & \sigma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} = \frac{\sigma^2}{n}$$

Linear Combinations of IID Random V

The results for expected values and variances of linear combinations hold for any probability distrubition with finite expected value and variance.

They enable us to determine the expected value and variance of any linear combination of random variables.

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They do not, however, tell us what the *distribution* of the linear combination is.

So, we cannot yet answer a question like:

"What is the probability that the average value of a vector of 8 IID random variables with mean 2 and variance 4 lies between 1 and 3?"

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So, we cannot yet answer a question like:

"What is the probability that the average value of a vector of 8 IID random variables with mean 2 and variance 4 lies between 1 and 3?"

This type of question comes up very frequently in statistics.

Linear Combinations of IID Random V

The following results will often allow us to answer these questions.

Theorem If the random vector X has a multivariate normal distribution with mean vector μ and variance-covariance matrix V,

$$X \sim N(\mu, V)$$

then any linear combination $\beta' X$ also has a normal distribution, with

$$\beta' X \sim N(\beta' \mu, \beta' V \beta)$$

Sample Means

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The most important special case occurs when the components of X are IID $N(\mu, \sigma)$ representing a rancom sample, and all elements of β are $1/n$.

In this case, $\beta' X$ is called the **sample mean**

$$\beta = \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} \quad \text{then} \quad E(\beta' X) = \beta' \mu = \mu$$

and

$$Var(\beta' X) = \beta'(\sigma^2 I)\beta = \frac{\sigma^2}{n}$$

Sample Means

Theorem If X is a vector of IID random variables each having a $N(\mu, \sigma)$ distribution, then the **sample mean** defined by

$$\overline{x} \;=\; \frac{1}{n}\sum_{i=1}^{n} X_i$$

has a normal distribution with mean μ and variance σ^2/n

Sample Means

Theorem If X is a vector of IID random variables each having a $N(\mu, \sigma)$ distribution, then the **sample mean** defined by

$$\overline{x} \;=\; \frac{1}{n}\sum_{i=1}^{n} X_i$$

has a normal distribution with mean μ and variance σ^2/n

In the usual notation, if the X_i are IID with

$$X_i \sim N\left(\mu, \sigma\right), \quad i = 1, 2, \ldots, n$$

then

$$\overline{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

Sample Means

Example SAT scores may be assumed to be normally distributed with mean $\mu = 500$ and standard deviation $\sigma = 100$.

What is the distribution of the mean $\overline{x}$ of a random sample of 400 SAT scores?

Sample Means

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What is the distribution of the mean $\overline{x}$ of a random sample of 400 SAT scores?

The theorem tells us that

$$\overline{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \;=\; N\left(500, \frac{100}{\sqrt{400}}\right) \;=\; N(500, 5)$$

Sample Means

Example Some IQ scores may be assumed to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

What is the distribution of the mean $\overline{x}$ of a random sample of 100 IQ scores?

Sample Means

Example Some IQ scores may be assumed to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

What is the distribution of the mean $\overline{x}$ of a random sample of 100 IQ scores?

The theorem tells us that

$$\overline{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \;=\; N\left(100, \frac{15}{\sqrt{100}}\right) \;=\; N(100, 1.5)$$

Sample Means

Example What is the probability that the mean of a random sample of size 5 from a $N(3, 2)$ distribution lies between 2 and 4?

Sample Means

Example What is the probability that the mean of a random sample of size 5 from a $N(3, 2)$ distribution lies between 2 and 4?

The theorem tells us that

$$\overline{x} \sim N\left(3, \frac{2}{\sqrt{5}}\right)$$

The mean is considered a single observation from this distribution, so the probability that it falls between 2 and 4 is:

`pnorm(4,3,2/sqrt(5))-pnorm(2,3,2/sqrt(5))` or 0.7364475

Sample Means

Example What is the probability that the mean of a random sample of size 10 from a $N(120, 25)$ distribution is less than 123?

Sample Means

Example What is the probability that the mean of a random sample of size 10 from a $N(120, 25)$ distribution is less than 123?

The theorem tells us that

$$\overline{x} \sim N\left(120, \frac{25}{\sqrt{10}}\right)$$

The mean is considered a single observation from this distribution, so the probability that it is less than 123 is:

`pnorm(123,120,25/sqrt(10))` or 0.6478318

Sample Means

Example What is the probability that the mean of a random sample of size 100 from a $N(500, 100)$ distribution is less than 495 or more than 505?

Sample Means

Example What is the probability that the mean of a random sample of size 100 from a $N(500, 100)$ distribution is less than 495 or more than 505?

The theorem tells us that

$$\overline{x} \sim N\left(500, \frac{100}{\sqrt{100}}\right)$$

The mean is considered a single observation from this distribution, so the probability that it falls between 495 and 505 is:

```
1-pnorm(505,500,100/sqrt(100))+pnorm(495,500,10
```
or 0.6170751

Sample Means

Example What is the probability that the mean of a random sample of size 200 from a $N(100, 15)$ distribution is more than 98?

Sample Means

Example What is the probability that the mean of a random sample of size 200 from a $N(100, 15)$ distribution is more than 98?

The theorem tells us that

$$\overline{x} \sim N\left(100, \frac{15}{\sqrt{200}}\right)$$

The mean is considered a single observation from this distribution, so the probability that it is more than 98 is:

`1-pnorm(98,100,15/sqrt(200))` or 0.9703268