Variance: The Bivariate Case

If X is a bivariate random variable,

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

then we need *three* parameters to fully specify the variance-covariance matrix of X:

- The variance of X_1
- The variance of X_2
- The covariance of X_1 and X_2

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The usual notation for these quantities is:

- The variance of X_1 is denoted by σ_1^2
- The variance of X_2 is denoted by σ_2^2
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$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

The matrix V is called the *variance-covariance* matrix of X

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To compute the expected value of Y, we needed to know β, the vector of coefficients, and μ, the vector of expected values.

To compute the variance of Y, we need to know V, the variance-covariance matrix of X, and β. The variance of Y is given by the matrix product

$$
\begin{aligned}
V(Y) &= \beta' V \beta \\
&= [\beta_1 \ \beta_2] \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}
\end{aligned}
$$

Variance: The Bivariate Case

In matrix notation, if X is a bivariate random variable with variance-covariance matrix V, and β a vector of coefficients,

$$\text{If} \quad Y = \beta' X \qquad \text{and} \qquad Var(X) = V$$

then

$$V(Y) = \beta' V \beta$$

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Many common statistical formulas can be written very compactly using quadratic forms. For example, the non-matrix version of the formula above is:

$$V(Y) = \sum_{i=1}^{2} \beta_i^2 \sigma_i^2 + 2 \sum_{i=1}^{2} \sum_{j>i} \beta_i \beta_j \sigma_{ij}$$

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$$= [3\ 5] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8$$

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$$= [2\ 4] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 6$$

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When all covariances are zero, the variance of a sum of random variables is the sum of their individual variances.

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If Y is the mean of X_1 and X_2, then $\beta' = [\frac{1}{2}\ \frac{1}{2}]$ and

$$V(Y) = \beta' V \beta = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{\sigma^2}{2}$$

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This is important because a sample of size n is usually treated as a vector of n random variables with zero covariances and common variance σ^2.

Variance: The Bivariate Case

In this case, β' has n components, and is given by

$$\beta' V \beta \ = \ \left[\frac{1}{n} \ \frac{1}{n} \cdots \ \frac{1}{n} \right]$$

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However, keep in mind that all of these special cases are covered by the general formula

$$Var(\beta' X) \; = \; \beta' V \beta$$

regardless of what β and the variance-covariance matrix V are.