The Bivariate Case

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Different values of β_1 and β_2 produce different linear combinations, including:

- The *sum* $X_1 + X_2$: $\quad \beta_1 = \beta_2 = 1$
- The *difference* $X_1 - X_2$: $\quad \beta_1 = 1, \; \beta_2 = -1$
- The *mean* $(X_1 + X_2)/2$: $\quad \beta_1 = \beta_2 = 1/2$

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- The *difference* $X_1 - X_2$: $\quad \beta_1 = 1,\ \beta_2 = -1$
- The *mean* $(X_1 + X_2)/2$: $\quad \beta_1 = \beta_2 = 1/2$

Less common linear combinations include $3X_1 - X_2$, $X_1 - 2X_2$, $X_2 - 2X_1$, and so on.

The Bivariate Case

There is a simple relationship between the expected value $E(Y)$ of a linear combination of two random variables. Let

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \mu$$

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Values of β for a few linear combinations of X_1 and X_2 are:

$$Y = X_1 + X_2 \qquad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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$$Y = X_1 + 2X_2 \qquad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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$$Y = \frac{1}{2}(X_1 + X_2) \qquad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

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$$Y = 3X_1 \qquad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

The Bivariate Case

Example: Two dice are rolled. The random variable X_1 represents the number showing on the first die, while X_2 is the number showing on the second. In a previous lecture, we established that when two dice are rolled,

$$\text{If} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{then} \quad E(X) = \mu = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$$

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$$\text{If} \quad Y = X_1 + X_2 \quad \text{then} \quad Y = \beta' X$$

where

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \beta' = [\beta_1 \ \beta_2] = [1 \ 1]$$

The Bivariate Case

In general, if the relationship between Y and X is:

$$Y = \beta' X \; = \; [1 \; 1] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \; = \; X_1 + X_2$$

Then the relationship between $E(Y)$ and $E(X)$ is:

$$E(Y) = \beta' E(X) \; = \; \beta' \mu \; = \; [1 \; 1] \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix} \; = \; 3.5 + 3.5 \; = \; 7$$

The Bivariate Case

Example: Two dice are rolled. The random variable X_1 represents the number showing on the first die, while X_2 is the number showing on the second. As before,

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$$\text{If} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{then} \quad E(X) = \mu = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$$

$$\text{If} \quad Y = X_1 - X_2 \quad \text{then} \quad Y = \beta' X$$

and

$$E(Y) = \beta' E(X) \; = \; \beta' \mu \; = \; [1 \;\; -1] \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix} \; = \; 3.5 - 3.5 \; = \; 0$$

The Bivariate Case

Example: A bivariate random variable X is defined as

$$\text{If} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where X_1 is a Poisson random variable with $\lambda_1 = 2$ and X_2 is a Poisson random variable with $\lambda_2 = 3$.

What is the expected value of the random variable $Y = 2X_1 - X_2$?

The Bivariate Case

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$$\text{If} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where X_1 is a Poisson random variable with $\lambda_1 = 2$ and X_2 is a Poisson random variable with $\lambda_2 = 3$.

What is the expected value of the random variable $Y = 2X_1 - X_2$?

From the properties of the Poisson distribution, its expected value is λ, so

$$E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The Bivariate Case

Then

$$Y = 2X_1 - X_2 \; = \; \beta' X \quad \text{where} \quad \beta \; = \; \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

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$$Y = 2X_1 - X_2 \ = \ \beta' X \quad \text{where} \quad \beta \ = \ \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

And

$$E(Y) = \beta' E(X) \ = \ \beta' \mu \ = \ [2 \ \ -1] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$= \ [2 \ \ -1] \begin{bmatrix} 2 \\ 3 \end{bmatrix} \ = \ 2 \cdot 2 + (-1) \cdot 3 \ = \ 1$$

The Bivariate Case

Then

$$Y = 2X_1 - X_2 \;=\; \beta' X \quad \text{where} \quad \beta \;=\; \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

And

$$E(Y) = \beta' E(X) \;=\; \beta' \mu \;=\; [2 \;\; -1] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$=\; [2 \;\; -1] \begin{bmatrix} 2 \\ 3 \end{bmatrix} \;=\; 2 \cdot 2 + (-1) \cdot 3 \;=\; 1$$

The expected value of Y is 1.

The Bivariate Case

Example: A bivariate random variable X is defined as

$$\text{If} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where X_1 is a Bernoulli random variable with probability of success $p_1 = 0.4$ and X_2 is a Bernoulli random variable with $p_2 = 0.5$.

What is the expected value of the random variable $Y = X_1 + X_2$?

The Bivariate Case

Example: A bivariate random variable X is defined as

$$\text{If} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where X_1 is a Bernoulli random variable with probability of success $p_1 = 0.4$ and X_2 is a Bernoulli random variable with $p_2 = 0.5$.

What is the expected value of the random variable $Y = X_1 + X_2$?

From the properties of the Bernoulli distribution, its expected value is the probability of success p, so

$$E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}$$

The Bivariate Case

Then

$$Y = X_1 + X_2 \;=\; \beta' X \quad \text{where} \quad \beta \;=\; \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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And

$$E(Y) = \beta' E(X) \;=\; \beta' \mu \;=\; [1\ 1] \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$=\; [1\ 1] \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix} \;=\; 1 \cdot 0.4 + 1 \cdot 0.5 \;=\; 0.9$$

The Bivariate Case

Then

$$Y = X_1 + X_2 \; = \; \beta' X \quad \text{where} \quad \beta \; = \; \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

And

$$E(Y) = \beta' E(X) \; = \; \beta' \mu \; = \; [1 \; 1] \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$= \; [1 \; 1] \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix} \; = \; 1 \cdot 0.4 + 1 \cdot 0.5 \; = \; 0.9$$

The expected value of Y is 0.9.

The Bivariate Case

Example: A bivariate random variable X is defined as

$$\text{If} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where X_1 is a Poisson random variable with parameter $\lambda = 4$ and X_2 is a geometric random variable with $p = 0.5$.

What is the expected value of the random variable $Y = X_1 + 2X_2$?

The Bivariate Case

Example: A bivariate random variable X is defined as

$$\text{If} \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where X_1 is a Poisson random variable with parameter $\lambda = 4$ and X_2 is a geometric random variable with $p = 0.5$.

What is the expected value of the random variable $Y = X_1 + 2X_2$?

From the properties of the Bernoulli distribution, its expected value is the probability of success p, so

$$E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} \lambda \\ (1-p)/p \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

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Then

$$Y = X_1 + 2X_2 \;=\; \beta' X \quad \text{where} \quad \beta \;=\; \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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And

$$E(Y) = \beta' E(X) \;=\; \beta' \mu \;=\; [1\ 2] \begin{bmatrix} \lambda \\ (1-p)/p \end{bmatrix}$$

$$=\; [1\ 2] \begin{bmatrix} 4 \\ 1 \end{bmatrix} \;=\; 1 \cdot 4 + 2 \cdot 1 \;=\; 6$$

The Bivariate Case

Then

$$Y = X_1 + 2X_2 \; = \; \beta' X \quad \text{where} \quad \beta \; = \; \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

And

$$E(Y) = \beta' E(X) \; = \; \beta' \mu \; = \; [1\ 2] \begin{bmatrix} \lambda \\ (1-p)/p \end{bmatrix}$$

$$= \; [1\ 2] \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \; 1 \cdot 4 + 2 \cdot 1 \; = \; 6$$

The expected value of Y is 6.

The Bivariate Case

In summary, if $Y = \beta' X$, where

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \quad \text{and} \quad E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \mu$$

then

$$E(Y) = \beta' E(X) = [\beta_1 \; \beta_2] \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \beta_1 \mu_1 + \beta_2 \mu_2$$

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then

$$E(Y) = \beta' E(X) = [\beta_1 \ \beta_2] \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \beta_1 \mu_1 + \beta_2 \mu_2$$

This equation holds regardless of the probability distributions of X_1 and X_2.

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This equation holds regardless of the probability distributions of X_1 and X_2.

The only requirement is that $E(X_1)$ and $E(X_2)$ must exist.