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We will now explore some results in combinatorics, starting with the "multiplication rule"

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This reflects the general idea that if you can do one task m ways and a second task n ways, the number of different ways you can do the two tasks is $m \times n$.

Counting Rules for two items

The rule applies to any ordered pair.

Suppose we are going to make an ordered pair

$$(P, Q)$$

with:

- P chosen from a set of n possibilities
- Q from a set of m possibilities

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the number of distinct ordered pairs that can possibly result is:

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The m times n rule says that we have a total of 6 possibilities. They are:

sedan manual	sedan automatic
hatchback manual	hatchback manual
convertible manual	convertible automatic

Counting Rules for n-tuples

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In general, if we are choosing an ordered list of k elements with n_1 choices for the first element, n_2 for the second, and so on, the number of possible ordered $k - tuples$ is:

$$n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

Permutations

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The number of permutations (ordered subsets) of size r taken from a set of n objects is:

$$_nP_r = \frac{n!}{(n-r)!}$$

Permutations

For the problem of picking three officers from a class of twenty four, this is:

$$_{24}R_3 = \frac{24!}{21!} = \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdots 2 \cdot 1}{21 \cdot 10 \cdots 2 \cdot 1}$$

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To compute the number of ways a class of 600 can elect three officers, use

=PERMUT(600,3) $\quad = \quad 214,921,200$

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In summary, a permutation is the number of ways to arrange of r out of n objects if:

- The n objects are distinct
- There is no repitition in the list
- Order is important

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Since the order matters, this is a permutation of 3 objects out of 22, and the formula is:

=PERMUT(22,3)=9,240

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Again the order matters, this is a permutation of 4 objects out of 26, and the formula is:

=PERMUT(26,4)=358,800

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Permutations with Nondistinct Items

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If the number of objects of kind i is n_i, $i = 1, 2, \ldots, k$ the number of arrangements (permutations) is:

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

where $n_1 + n_2 + \cdots + n_k = n$