Completing the Square

Completing the square is an algebraic technique for converting a standard quadratic expression

$$ax^2 + bx + c$$

into one of the form

$$(\alpha x + \beta)^2 + \gamma$$

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We start by equating the two expressions:

$$ax^2 + bx + c \;=\; (\alpha x + \beta)^2 + \gamma \;=\; \alpha^2 x^2 + 2\alpha\beta x + \beta^2 + \gamma$$

Completing the Square

$$ax^2 + bx + c \;=\; \alpha^2 x^2 + 2\alpha\beta x + \beta^2 + \gamma$$

Two polynomials are equal if and only if their coefficients are the same for each power of x.

Completing the Square

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Two polynomials are equal if and only if their coefficients are the same for each power of x.

That means we can write three equations, one for each power of x:

- $a = \alpha^2$
- $b = 2\alpha\beta$
- $c = \beta^2 + \gamma$

Completing the Square

Starting with the easiest one, we first solve for α:

$$a = \alpha^2 \quad \text{so} \quad \alpha = \sqrt{a}$$

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$$b = 2\alpha\beta \quad \text{so} \quad \beta = \frac{b}{2\alpha} = \frac{b}{2\sqrt{a}}$$

Finally

$$c = \beta^2 + \gamma \quad \text{so} \quad \gamma = c - \beta^2 = c - \frac{b^2}{4a}$$

Completing the Square

In summary, if

$$ax^2 + bx + c \;=\; (\alpha x + \beta)^2 + \gamma$$

then:

- $\alpha = \sqrt{a}$
- $\beta = \frac{b}{2\sqrt{a}}$
- $\gamma = c - \frac{b^2}{4a}$

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Note: If $a < 0$, first write the quadratic as

$$-(ax^2 + bx + c) \;=\; -[(\alpha x + \beta)^2 + \gamma]$$

Question 1

Write the quadratic

$$x^2 + 8x + 5 \quad \text{in the form} \quad (\alpha x + \beta)^2 + \gamma$$

1. $(x-4)^2 + 9$
2. $(x+2)^2 - 3$
3. $(x+4)^2 - 9$
4. $(x-4)^2 - 9$
5. $(x+2)^2 - 9$
6. none of the above

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4. $(x-4)^2 - 9$
2. $(x+2)^2 - 3$
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3. $(x+4)^2 - 9$
6. none of the above

3. $(x+4)^2 - 9$

Completing the Square

We want to find α, β, and γ so that:

$$x^2 + 8x + 5 \;=\; (\alpha x + \beta)^2 + \gamma$$

then since $a = 1$, $b = 8$, and $c = 5$,

- $\alpha = \sqrt{a} = 1$
- $\beta = \frac{b}{2\sqrt{a}} = \frac{8}{2\sqrt{1}} = 4$
- $\gamma = c - \frac{b^2}{4a} = 5 - \frac{8^2}{4 \cdot 1} = 5 - 16 = -11$

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One can easily check that

$$x^2 + 8x + 5 \;=\; (x + 4)^2 - 11$$

Question 2

Write the quadratic

$$x^2 + x + 1 \quad \text{in the form} \quad \alpha x + \beta)^2 + \gamma$$

1. $(x + \frac{1}{2})^2 - \frac{3}{4}$
2. $(x + \frac{1}{2})^2 + \frac{3}{4}$
3. $(x - \frac{1}{2})^2 - \frac{3}{4}$
4. $(x - \frac{1}{2})^2 + \frac{1}{4}$
5. $(x + \frac{1}{2})^2 + \frac{1}{4}$
6. none of the above

Question 2

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$$x^2 + x + 1 \quad \text{in the form} \quad \alpha x + \beta)^2 + \gamma$$

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2. $(x + \frac{1}{2})^2 + \frac{3}{4}$
3. $(x - \frac{1}{2})^2 - \frac{3}{4}$
4. $(x - \frac{1}{2})^2 + \frac{1}{4}$
5. $(x + \frac{1}{2})^2 + \frac{1}{4}$
6. none of the above

2. $(x + \frac{1}{2})^2 + \frac{3}{4}$

Completing the Square

We want to find α, β, and γ so that:

$$x^2 + x + 1 \;=\; (\alpha x + \beta)^2 + \gamma$$

then since $a = 1$, $b = 1$, and $c = 1$,

- $\alpha = \sqrt{a} = 1$
- $\beta = \frac{b}{2\sqrt{a}} = \frac{1}{2\sqrt{1}} = \frac{1}{2}$
- $\gamma = c - \frac{b^2}{4a} = 1 - \frac{1^2}{4 \cdot 1} = \frac{3}{4}$

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We want to find α, β, and γ so that:

$$x^2 + x + 1 \ = \ (\alpha x + \beta)^2 + \gamma$$

then since $a = 1$, $b = 1$, and $c = 1$,

- $\alpha = \sqrt{a} = 1$
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One can easily check that

$$x^2 + x + 1 \ = \ \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

Completing the Square

The technique of completing the square is useful in many situations, but we are interested in turning a rational function of the form

$$\frac{A}{ax^2 + bx + c}$$

where $ax^2 + bx + c$ is irreducible (i.e., $b^2 - 4ac < 0$), into an expression of the form

$$\frac{A}{(\alpha x + \beta)^2 + \gamma}$$

Completing the Square

This allows us to use the formula

$$\int \frac{dz}{z^2 + d^2} \; = \; \frac{1}{d} \tan^{-1}\left(\frac{z}{d}\right) + C$$

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Here we identify

$$z = (\alpha x + \beta) \quad \text{and} \quad d = \sqrt{\gamma}$$