Completing the square is an algebraic technique for converting a standard quadratic expression

$$ax^2 + bx + c$$

into one of the form

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We start by equating the two expressions:

$$ax^{2} + bx + c = (\alpha x + \beta)^{2} + \gamma = \alpha^{2}x^{2} + 2\alpha\beta x + \beta^{2} + \gamma$$

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That means we can write three equations, one for each power of *x*:

- $a = \alpha^2$
- $b = 2\alpha\beta$
- $c = \beta^2 + \gamma$

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 so $\alpha = \sqrt{a}$

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Finally

$$c = \beta^2 + \gamma$$
 so $\gamma = c - \beta^2 = c - \frac{b^2}{4a}$

In summary, if

$$ax^2 + bx + c = (\alpha x + \beta)^2 + \gamma$$

then:

• $\alpha = \sqrt{a}$ • $\beta = \frac{b}{2\sqrt{a}}$ • $\gamma = c - \frac{b^2}{4a}$

In summary, if

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then:

•
$$\alpha = \sqrt{a}$$

• $\beta = \frac{b}{2\sqrt{a}}$
• $\gamma = c - \frac{b^2}{4a}$

Note: If a < 0, first write the quadratic as

$$-(ax^{2} + bx + c) = -[(\alpha x + \beta)^{2} + \gamma]$$

Write the quadratic

$$x^2 + 8x + 5$$
 in the form $(\alpha x + \beta)^2 + \gamma$

1.
$$(x-4)^2 + 9$$
 4. $(x-4)^2 - 9$

2.
$$(x+2)^2 - 3$$
 5. $(x+2)^2 - 9$

3. $(x+4)^2 - 9$ 6. none of the above

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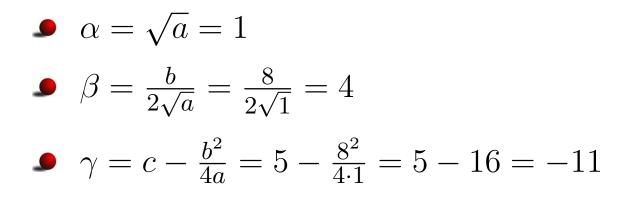
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We want to find α , β , and γ so that:

$$x^2 + 8x + 5 = (\alpha x + \beta)^2 + \gamma$$

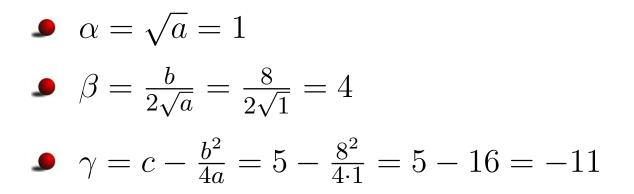
then since a = 1, b = 8, and c = 5,



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One can easily check that

$$x^2 + 8x + 5 = (x+4)^2 - 11$$

Write the quadratic

 $x^2 + x + 1$ in the form $\alpha x + \beta)^2 + \gamma$

1.
$$(x+\frac{1}{2})^2 - \frac{3}{4}$$
 4. $(x-\frac{1}{2})^2 + \frac{1}{4}$

2.
$$(x+\frac{1}{2})^2 + \frac{3}{4}$$
 5. $(x+\frac{1}{2})^2 + \frac{1}{4}$

3. $(x - \frac{1}{2})^2 - \frac{3}{4}$ 6. none of the above

Write the quadratic

 $x^2 + x + 1$ in the form $\alpha x + \beta)^2 + \gamma$

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$$(x+\frac{1}{2})^2 - \frac{3}{4}$$
 4. $(x-\frac{1}{2})^2 + \frac{1}{4}$

2.
$$(x+\frac{1}{2})^2 + \frac{3}{4}$$

5.
$$(x+\frac{1}{2})^2+\frac{1}{4}$$

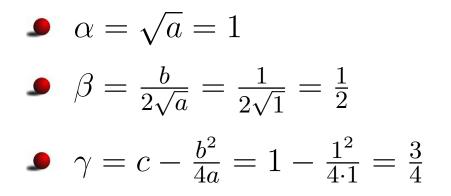
3. $(x - \frac{1}{2})^2 - \frac{3}{4}$ 6. none of the above

2. $(x+\frac{1}{2})^2 + \frac{3}{4}$

We want to find α , β , and γ so that:

$$x^2 + x + 1 = (\alpha x + \beta)^2 + \gamma$$

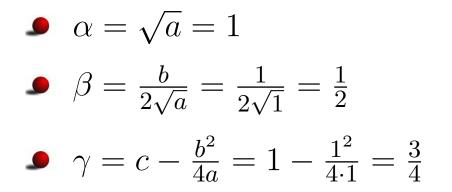
then since a = 1, b = 1, and c = 1,



We want to find α , β , and γ so that:

$$x^2 + x + 1 = (\alpha x + \beta)^2 + \gamma$$

then since a = 1, b = 1, and c = 1,



One can easily check that

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

The technique of completing the square is useful in many situations, but we are interested in turning a rational function of the form

$$\frac{A}{ax^2 + bx + c}$$

where $ax^2 + bx + c$ is irreducible (i.e., $b^2 - 4ac < 0$), into an expression of the form

$$\frac{A}{(\alpha x + \beta)^2 + \gamma}$$

This allows us to use the formula

$$\int \frac{dz}{z^2 + d^2} = \frac{1}{d} \tan^{-1} \left(\frac{z}{d}\right) + C$$

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Here we identify

$$z = (\alpha x + \beta)$$
 and $d = \sqrt{\gamma}$