Trigonometric Identities

It is possible to fill a book with trigonometric identities.

However, by far the most important is:

$$\cos^2 x + \sin^2 x = 1$$

Trigonometric Identities

It is possible to fill a book with trigonometric identities.

However, by far the most important is:

$$\cos^2 x + \sin^2 x = 1$$

This gives rise to the expressions

$$\cos^2 x = 1 - \sin^2 x \quad \text{and} \quad \sin^2 x = 1 - \cos^2 x$$

Trigonometric Identities

It is possible to fill a book with trigonometric identities.

However, by far the most important is:

$$\cos^2 x + \sin^2 x = 1$$

This gives rise to the expressions

$$\cos^2 x = 1 - \sin^2 x \quad \text{and} \quad \sin^2 x = 1 - \cos^2 x$$

From these we get

$$\cos x = \sqrt{1 - \sin^2 x} \quad \text{and} \quad \sin x = \sqrt{1 - \cos^2 x}$$

Trigonometric Identities

$$\cos x = \sqrt{1 - \sin^2 x} \quad \text{and} \quad \sin x = \sqrt{1 - \cos^2 x}$$

The above expressions can be used to turn any expression involving $\cos x$ into an equivalent in terms of $\sin x$, and vice-versa.

Trigonometric Identities

$$\cos x = \sqrt{1 - \sin^2 x} \quad \text{and} \quad \sin x = \sqrt{1 - \cos^2 x}$$

The above expressions can be used to turn any expression involving $\cos x$ into an equivalent in terms of $\sin x$, and vice-versa.

One reason you might need to do this is to evaluate an integral like

$$\int \sin^3 x$$

Trigonometric Identities

Using the identity $\sin^2 x = 1 - \cos^2 x$, we can write the integral as

$$\int \sin^3 x \; = \; \int (1 - \cos^2 x) \sin x \; dx$$

Trigonometric Identities

Using the identity $\sin^2 x = 1 - \cos^2 x$, we can write the integral as

$$\int \sin^3 x \ = \ \int (1 - \cos^2 x) \sin x \, dx$$

If we let $u = \cos x$, then

$$\frac{du}{dx} = -\sin x \quad \text{so} \quad -du = \sin x \, dx$$

Trigonometric Identities

Using the identity $\sin^2 x = 1 - \cos^2 x$, we can write the integral as

$$\int \sin^3 x \ = \ \int (1 - \cos^2 x) \sin x \ dx$$

If we let $u = \cos x$, then

$$\frac{du}{dx} = -\sin x \quad \text{so} \quad -du = \sin x \ dx$$

Now on substitution the integral becomes

$$\int \sin^3 dx = \int (1 - \cos^2 x) \sin x \ dx = -\int (1 - u^2) \ du$$

Trigonometric Identities

The result is:

$$\int \sin^3 dx \;=\; -\int (1-u^2)\,du \;=\; \frac{u^3}{3} - u$$

$$=\; \frac{\cos^3 x}{3} - \cos x + C$$

Question 1

Evaluate the integral

$$\int \cos^3 x \, dx$$

1. $\sin x - (1/3)\sin^3 x + C$
2. $\sin x + (1/3)\sin^3 x + C$
3. $\sin^3 x - (1/3)\sin x + C$
4. $\sin^3 x + (1/3)\sin x + C$
5. $3\sin x - \sin^3 x + C$
6. none of the above

Question 1

Evaluate the integral

$$\int \cos^3 x \, dx$$

1. $\sin x - (1/3)\sin^3 x + C$
2. $\sin x + (1/3)\sin^3 x + C$
3. $\sin^3 x - (1/3)\sin x + C$
4. $\sin^3 x + (1/3)\sin x + C$
5. $3\sin x - \sin^3 x + C$
6. none of the above

1. $\sin x - (1/3)\sin^3 x + C$

Question 2

Evaluate the integral

$$\int \sin^3 x \cos^2 x \, dx$$

1. $(1/5)\cos^5 x + (1/3)\cos^3 x + C$
2. $(1/3)\cos^3 x - (1/5)\cos^5 x + C$
3. $(1/5)\sin^5 x - (1/3)\sin^3 x + C$
4. $(1/5)\cos^5 x - (1/3)\cos^3 x + C$
5. $5\cos^5 x - 3\cos^3 x + C$
6. none of the above

Question 2

Evaluate the integral

$$\int \sin^3 x \cos^2 x \, dx$$

1. $(1/5)\cos^5 x + (1/3)\cos^3 x + C$
2. $(1/3)\cos^3 x - (1/5)\cos^5 x + C$
3. $(1/5)\sin^5 x - (1/3)\sin^3 x + C$
4. $(1/5)\cos^5 x - (1/3)\cos^3 x + C$
5. $5\cos^5 x - 3\cos^3 x + C$
6. none of the above

4. $(1/5)\cos^5 x - (1/3)\cos^3 x + C$

Trigonometric Identities

In general, this technique is useful for integrals of the form

$$\int \sin^m x \cos^n x \; dx$$

Trigonometric Identities

In general, this technique is useful for integrals of the form

$$\int \sin^m x \cos^n x \, dx$$

There are three special cases to consider, depending on whether m and n are even or odd.

Trigonometric Identities

$$\int \sin^m x \cos^n x \, dx$$

Case 1: m is odd, that is, $m = 2k + 1$ for some natural number k

Trigonometric Identities

$$\int \sin^m x \cos^n x \; dx$$

Case 1: m is odd, that is, $m = 2k + 1$ for some natural number k

In this case write the integral as

$$\int \sin^{2k+1} x \cos^n x \; dx \;=\; \int \sin^{2k} x \cos^n x \;\; \sin x \; dx$$

Trigonometric Identities

$$\int \sin^m x \cos^n x \, dx$$

Case 1: m is odd, that is, $m = 2k + 1$ for some natural number k

In this case write the integral as

$$\int \sin^{2k+1} x \cos^n x \, dx \; = \; \int \sin^{2k} x \cos^n x \; \sin x \, dx$$

Now substitute

$$(1 - \cos^2 x)^k \quad \text{for} \quad \sin^{2k}$$

Trigonometric Identities

The result is:

$$\int (1 - \cos^2 x)^k \cos^n x \ \sin x \ dx$$

Trigonometric Identities

The result is:

$$\int (1-\cos^2 x)^k \cos^n x \ \sin x \ dx$$

Now let

$$u = \cos x \quad \text{so} \quad \frac{du}{dx} = -\sin x \ \text{ and } \ -du = \sin x \ dx$$

Trigonometric Identities

The result is:

$$\int (1-\cos^2 x)^k \cos^n x \; \sin x \; dx$$

Now let

$$u = \cos x \quad \text{so} \quad \frac{du}{dx} = -\sin x \; \text{ and } \; -du = \sin x \; dx$$

On substitution the integral becomes

$$\int \sin^m x \cos^n x \; dx \; = \; -\int (1-u^2)^k u^n \; du$$

Trigonometric Identities

The result is:

$$\int (1-\cos^2 x)^k \cos^n x \ \sin x \ dx$$

Now let

$$u = \cos x \quad \text{so} \quad \frac{du}{dx} = -\sin x \ \text{ and } \ -du = \sin x \ dx$$

On substitution the integral becomes

$$\int \sin^m x \cos^n x \ dx \ = \ -\int (1-u^2)^k u^n \ du$$

This can be either expanded and integrated as a polynomial, or integrated by parts.

Trigonometric Identities

$$\int \sin^m x \cos^n x \, dx$$

Case 2: n is odd, that is, $n = 2k + 1$ for some natural number k

Trigonometric Identities

$$\int \sin^m x \cos^n x \, dx$$

Case 2: n is odd, that is, $n = 2k + 1$ for some natural number k

In this case write the integral as

$$\int \sin^m x \cos^{2k+1} x \, dx \;=\; \int \sin^m x \cos^{2k} x \;\cos x \, dx$$

Trigonometric Identities

$$\int \sin^m x \cos^n x \; dx$$

Case 2: n is odd, that is, $n = 2k + 1$ for some natural number k

In this case write the integral as

$$\int \sin^m x \cos^{2k+1} x \; dx \;=\; \int \sin^m x \cos^{2k} x \;\cos x \; dx$$

Now substitute

$$(1 - \sin^2 x)^k \quad \text{for} \quad \cos^{2k}$$

Trigonometric Identities

The result is:

$$\int \sin^m x(1 - \sin^2 x)^k \cos x \, dx$$

Trigonometric Identities

The result is:

$$\int \sin^m x(1 - \sin^2 x)^k \cos x \, dx$$

Now let

$$u = \sin x \quad \text{so} \quad \frac{du}{dx} = \cos x \text{ and } du = \cos x \, dx$$

Trigonometric Identities

The result is:

$$\int \sin^m x(1 - \sin^2 x)^k \cos x \, dx$$

Now let

$$u = \sin x \quad \text{so} \quad \frac{du}{dx} = \cos x \ \text{ and } \ du = \cos x \, dx$$

On substitution the integral becomes

$$\int \sin^m x \cos^n x \, dx \; = \; -\int u^m (1 - u^2)^k \, du$$

Trigonometric Identities

The result is:

$$\int \sin^m x(1 - \sin^2 x)^k \cos x \; dx$$

Now let

$$u = \sin x \quad \text{so} \quad \frac{du}{dx} = \cos x \;\; \text{and} \;\; du = \cos x \; dx$$

On substitution the integral becomes

$$\int \sin^m x \cos^n x \; dx \; = \; -\int u^m (1 - u^2)^k \; du$$

This can be either expanded and integrated as a polynomial, or integrated by parts.

Trigonometric Identities

Case 3: An integral of the following form with m and n both even:

$$\int \sin^m x \cos^n x \, dx$$

Trigonometric Identities

Case 3: An integral of the following form with m and n both even:

$$\int \sin^m x \cos^n x \; dx$$

Let's start with a simple case such as

$$\int \sin^2 x \, dx$$

Trigonometric Identities

Case 3: An integral of the following form with m and n both even:

$$\int \sin^m x \cos^n x \, dx$$

Let's start with a simple case such as

$$\int \sin^2 x \, dx$$

The key to handling this type of integral are the trigonometric identities for $\sin 2\theta$ and $\cos 2\theta$

Trigonometric Identities

The following famous equation known as **Euler's Identity** ties together the exponential function, complex numbers, and trig functions:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Trigonometric Identities

The following famous equation known as **Euler's Identity** ties together the exponential function, complex numbers, and trig functions:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For the double angle formulas, consider

$$\cos 2\theta + i\sin 2\theta = e^{i\cdot 2\theta} = e^{i\theta}\cdot e^{i\theta}$$

Trigonometric Identities

The following famous equation known as **Euler's Identity** ties together the exponential function, complex numbers, and trig functions:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For the double angle formulas, consider

$$\cos 2\theta + i\sin 2\theta = e^{i\cdot 2\theta} = e^{i\theta}\cdot e^{i\theta}$$

Then

$$e^{i\theta}\cdot e^{i\theta} = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$

Trigonometric Identities

Expanding

$$e^{i\theta} \cdot e^{i\theta} = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$

we get

$$(\cos^2\theta + i^2\sin^2\theta) + i(2\sin\theta\cos\theta)$$

Trigonometric Identities

Expanding

$$e^{i\theta} \cdot e^{i\theta} = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$

we get

$$(\cos^2\theta + i^2\sin^2\theta) + i(2\sin\theta\cos\theta)$$

Replacing i^2 by -1 gives

$$\cos 2\theta + i\sin 2\theta = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$$

Trigonometric Identities

Expanding

$$e^{i\theta} \cdot e^{i\theta} = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$

we get

$$(\cos^2\theta + i^2\sin^2\theta) + i(2\sin\theta\cos\theta)$$

Replacing i^2 by -1 gives

$$\cos 2\theta + i\sin 2\theta = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$$

In an equality involving complex numbers, the real parts must be equal. Equating the real parts gives us the identity we want:

$$\cos 2\theta \;=\; \cos^2\theta - \sin^2\theta$$

Trigonometric Identities

Now replace $\cos^2\theta$ by $1-\sin^2\theta$ to get

$$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta \;=\; 1 - 2\sin^2\theta$$

Trigonometric Identities

Now replace $\cos^2\theta$ by $1 - \sin^2\theta$ to get

$$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta \ = \ 1 - 2\sin^2\theta$$

Solving for $\sin^2\theta$ gives us the form we will use:

$$\sin^2\theta \ = \ \frac{1}{2}\,(1 - \cos 2\theta)$$

Trigonometric Identities

Now replace $\cos^2\theta$ by $1 - \sin^2\theta$ to get

$$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta \;=\; 1 - 2\sin^2\theta$$

Solving for $\sin^2\theta$ gives us the form we will use:

$$\sin^2\theta \;=\; \frac{1}{2}(1 - \cos 2\theta)$$

Now we can substitute in our integrand:

$$\int \sin^2 x\,dx \;=\; \int \frac{1}{2}(1 - \cos 2x)\;dx \;=\; \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

Trigonometric Identities

Alternatively we could replace $\sin^2\theta$ by $1 - \cos^2\theta$

$$\cos 2\theta = \cos^2\theta - (1 - \cos^2\theta) \ = \ 2\cos^2\theta - 1$$

Trigonometric Identities

Alternatively we could replace $\sin^2\theta$ by $1-\cos^2\theta$

$$\cos 2\theta = \cos^2\theta - (1-\cos^2\theta) \;=\; 2\cos^2\theta - 1$$

Solving for $\cos^2\theta$ gives us the form we will use:

$$\cos^2\theta \;=\; \frac{1}{2}(1+\cos 2\theta)$$

Trigonometric Identities

Alternatively we could replace $\sin^2\theta$ by $1-\cos^2\theta$

$$\cos 2\theta = \cos^2\theta - (1-\cos^2\theta) \;=\; 2\cos^2\theta - 1$$

Solving for $\cos^2\theta$ gives us the form we will use:

$$\cos^2\theta \;=\; \frac{1}{2}\,(1+\cos 2\theta)$$

Now we can substitute in the integrand:

$$\int \cos^2 x\,dx \;=\; \int \frac{1}{2}\,(1+\cos 2x)\;dx \;=\; \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C$$

Trigonometric Identities

In general given an integrand of the following form with m and n even,

$$\int \sin^m x \cos^n x \, dx$$

write the integrand in the form

$$\int \sin^{2j} x \cos^{2k} x \, dx \quad \text{where } m = 2j, \; n = 2k$$

Trigonometric Identities

In general given an integrand of the following form with m and n even,

$$\int \sin^m x \cos^n x \; dx$$

write the integrand in the form

$$\int \sin^{2j} x \cos^{2k} x \; dx \quad \text{where } m = 2j, \; n = 2k$$

The result is:

$$\int \frac{1}{2}(1 - \cos 2x)^j \frac{1}{2}(1 + \cos 2x)^k \; dx$$

Trigonometric Identities

In general given an integrand of the following form with m and n even,

$$\int \sin^m x \cos^n x \, dx$$

write the integrand in the form

$$\int \sin^{2j} x \cos^{2k} x \, dx \quad \text{where } m = 2j, \; n = 2k$$

The result is:

$$\int \frac{1}{2}(1 - \cos 2x)^j \frac{1}{2}(1 + \cos 2x)^k \, dx$$

This can be either expanded and integrated as a polynomial, or integrated by parts.

Question 3

Evaluate the integral

$$\int \cos^2 x \, dx$$

1. $(1/2)\cos x \sin x - (1/2)x + C$
2. $2\cos x \sin x + 2x + C$
3. $(1/4)\sin 2x + (1/2)x + C$
4. $(1/2)\cos x (1/2)\sin x + C$
5. $(1/2)\sin x - (1/2)\cos x + C$
6. none of the above

Question 3

Evaluate the integral

$$\int \cos^2 x \, dx$$

1. $(1/2)\cos x \sin x - (1/2)x + C$
2. $2\cos x \sin x + 2x + C$
3. $(1/4)\sin 2x + (1/2)x + C$
4. $(1/2)\cos x(1/2)\sin x + C$
5. $(1/2)\sin x - (1/2)\cos x + C$
6. none of the above

3. $(1/4)\sin 2x + (1/2)x + C$

Question 4

Evaluate the integral

$$\int \sin^2 2x \, dx$$

1. $(1/2)x - (1/2)\sin 2x + C$
2. $(1/2)x - (1/8)\sin 2x + C$
3. $(1/2)x - (1/4)\sin 2x + C$
4. $(1/2)x + (1/8)\sin 4x + C$
5. $(1/2)x - (1/8)\sin 4x + C$
6. none of the above

Question 4

Evaluate the integral

$$\int \sin^2 2x \, dx$$

1. $(1/2)x - (1/2)\sin 2x + C$
2. $(1/2)x - (1/8)\sin 2x + C$
3. $(1/2)x - (1/4)\sin 2x + C$
4. $(1/2)x + (1/8)\sin 4x + C$
5. $(1/2)x - (1/8)\sin 4x + C$
6. none of the above

5. $(1/2)x - (1/8)\sin 4x + C$