Integration by Parts

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gave rise to the substitution rule for integration.

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Integrating both sides produces

$$f(g(x)) = \int f'(g(x))g'(x)\, dx$$

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Integrating both sides produces

$$fg = \int fg'dx + \int gf'dx$$

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As with the substitution rule, the integrand has to have a specific form.

When it does, we can substitute the expression on the right for the integral on the left.

Hopefully, the integral on the right is easier to evaluate than the one on the left.

Example 1

Consider the integral

$$\int x \cos x \; dx$$

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We want to identify f and g corresponding to the left hand side of:

$$\int f g' dx = f g - \int g f' dx$$

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We want to identify f and g corresponding to the left hand side of:

$$\int f g' dx = f g - \int g f' dx$$

If we let $f(x) = x$ and $g(x) = \sin x$, we have:

$$f(x) = x \qquad f'(x) = 1$$
$$g(x) = \sin x \qquad g'(x) = \cos x$$

Example 1

Now we substitute these values:

$$f(x) = x \qquad f'(x) = 1$$
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The result is:

$$\int x \cos x \; dx = x \sin x - \int \sin x \; dx$$

The integral on the right is $-\cos x$

Example 1

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The technique worked because

$\int \sin x \; dx$ is easier to evaluate than

$$\int x \cos x \; dx$$

Example 2

Now consider

$$\int xe^x \, dx$$

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We want to identify this integral as

$$\int f g' dx$$

for some functions f and g.

Example 2

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$$\int g f' \, dx$$

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In this case, good choices are:

$$f(x) = x \quad \text{and} \quad g(x) = e^x$$

Example 2

Then $g' = e^x$, $f' = 1$, and the integration by parts formula

$$\int fg' dx \;=\; fg - \int gf' \, dx$$

becomes

$$\int xe^x \, dx \;=\; xe^x - \int e^x \cdot 1 \, dx$$

$$=\; xe^x - e^x$$

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becomes

$$\int xe^x\,dx \;=\; xe^x - \int e^x \cdot 1\,dx$$
$$=\; xe^x - e^x$$

Again, the integral on the right is easier to evaluate than the one on the left.

Example 3

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Let $f(x) = x^2$ and $g(x) = g'(x) = e^x$. Then the integration by parts formula gives:

$$\int x^2 e^x \, dx \; = \; x^2 e^x - \int e^x \cdot 2x \, dx$$

$$= \; x^2 e^x - 2 \int x e^x \, dx$$

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As we saw, the integral on the right is $xe^x - e^x$.

Example 3

With this substitution, the result is

$$\int x^2 e^x \, dx \;=\; x^2 e^x - 2(xe^x - e^x)$$

or

$$\int x^2 e^x \, dx \;=\; x^2 e^x - 2xe^x + 2e^x$$

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Depending on the integrand, more than two applications of the integration by parts formula may be required.

The Tabular Method

A simple computational algorithm exists for repeated integration by parts.

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Suppose we want to evaluate an integral of the form

$$\int f(x)g(x)\, dx$$

Identify f and g, then create a table of the form:

$+$	g	$\int f dx$	$+g\int f dx$
$-$	g'	$\int\int f(dx)^2$	$-g'\int\int f(dx)^2$
$+$	g''	$\int\int\int f(dx)^3$	$g''\int\int\int f(dx)^3$
$-$	g'''	$\int\int\int\int f(dx)^4$	$g'''\int\int\int\int f(dx)^4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The Tabular Method

$+$	g	$\int f\,dx$	$+g\int f\,dx$
$-$	g'	$\int\int f(dx)^2$	$-g'\int\int f(dx)^2$
$+$	g''	$\int\int\int f(dx)^3$	$+g''\int\int\int f(dx)^3$
$-$	g'''	$\int\int\int\int f(dx)^4$	$-g'''\int\int\int\int f(dx)^4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- The left column of the table starts with $+$ and alternates signs
- The next column contains succesive derivatives of g
- The next column contains successive integrals of f
- The last column contains the product of the second and third columns.

The Tabular Method

$$\int x^2 e^x \, dx$$

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Identify $f = e^x$ and $g = x^2$, and create the table:

$+$	x^2	$\int e^x dx$	$+x^2e^x$
$-$	$2x$	$\int\int e^x (dx)^2$	$-2xe^x$
$+$	2	$\int\int\int e^x (dx)^3$	$+2e^x$
$-$	0	$\int\int\int\int e^x (dx)^4$	0

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$-$	0	$\int \int \int \int e^x (dx)^4$	0

Adding the entries in the rightmost column we obtain:

$$x^2e^x - 2xe^x + 2e^x$$

The Tabular Method

Now consider $\displaystyle\int x^3 \sin x \, dx$

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The table is:

$+$	x^3	$-\cos x$	$-x^3 \cos x$
$-$	$3x^2$	$-\sin x$	$+3x^2 \sin x$
$+$	$6x$	$\cos x$	$+6x \cos x$
$-$	6	$\sin x$	$-6 \sin x$
$+$	0	$-\cos x$	0

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$-$	6	$\sin x$	$-6 \sin x$
$+$	0	$-\cos x$	0

Adding the entries in the rightmost column we obtain:

$$-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$$

Question 1

Evaluate the integral

$$\int xe^{-x}dx$$

1. $xe^{-x} + e^{-x}$
2. $-xe^{-x} - e^{-x}$
3. $-xe^{-x} + e^{-x}$
4. $xe^{-x} - e^{x}$
5. $-xe^{x} + e^{-x}$
6. none of the above

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$$-xe^{-x} - e^{-x}$$

Question 2

Evaluate the integral

$$\int x^4 \cos x \, dx$$

1. $x^4 \sin x - 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x$
2. $x^4 \sin x + 4x^3 \cos x + 12x^2 \sin x - 24x \cos x + 24 \sin x$
3. $x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x$
4. $x^4 \sin x - 4x^3 \cos x + 12x^2 \sin x - 24x \cos x - 24 \sin x$
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$-$	$24x$	$\cos x$	$-24x \cos x$
$+$	24	$\sin x$	$24 \sin x$

Adding the entries in the rightmost column we obtain:

$$(x^4 - 12x^2 + 24) \sin x + (4x^3 - 24x) \cos x$$

Question 3

Evaluate the integral

$$\int \frac{x+2}{\sqrt[3]{2x+1}}\,dx$$

1. $\frac{3}{4}(x+2)(2x+1)^{2/3} - \frac{9}{40}(2x+1)^{5/3}$
2. $-\frac{3}{4}(x+2)(2x+1)^{2/3} + \frac{9}{40}(2x+1)^{5/3}$
3. $-\frac{3}{4}(x+2)(2x+1)^{2/3} - \frac{9}{40}(2x+1)^{5/3}$
4. $\frac{1}{4}(x+2)(2x+1)^{2/3} + \frac{6}{40}(2x+1)^{5/3}$
5. $-\frac{1}{4}(x+2)(2x+1)^{2/3} - \frac{6}{40}(2x+1)^{5/3}$
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Write the integrand as:

$$(x+2)(2x+1)^{-1/3}$$

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The table is:

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Adding the entries in the rightmost column we obtain:

$$\frac{3}{4}(x+2)(2x+1)^{2/3} - \frac{9}{40}(2x+1)^{5/3}$$

The Tabular Method - Remainders

In all of the examples of the tabular method so far, the process of building the table stopped when the next entry in the derivatives column became zero (meaning all the entries in the last column would be zero from that row on).

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Actually we can stop at any point, even if the next derivative is not zero.

If the next derivative is not zero, the final expression will contain an integral remainder term.

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Actually we can stop at any point, even if the next derivative is not zero.

If the next derivative is not zero, the final expression will contain an integral remainder term.

The remainder term is always the integral of the product of:

- The *next* entry in the derivative column (with the appropriate sign from the first column)
- The *current* entry in the integral column

The Tabular Method - Remainders

Example: Carry out the tabular method to evaluate

$$\int \frac{e^{-x}}{x}\,dx$$

using only the first two rows of the table and the remainder.

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The first two rows of the table are:

$+$	x^{-1}	$-e^{-x}$	$-x^{-1}e^{-x}$
$-$	$-x^{-2}$	e^{-x}	$x^{-2}e^{-x}$

The Tabular Method - Remainders

Now fill in the first two columns of the next row:

$+$	x^{-1}	$-e^{-x}$	$-x^{-1}e^{-x}$
$-$	$-x^{-2}$	e^{-x}	$x^{-2}e^{-x}$
$+$	$2x^{-3}$		

The Tabular Method - Remainders

Now fill in the first two columns of the next row:

$+$	x^{-1}	$-e^{-x}$	$-x^{-1}e^{-x}$
$-$	$-x^{-2}$	e^{-x}	$x^{-2}e^{-x}$
$+$	$2x^{-3}$		

The remainder term is the third entry in the derivative column times the second entry in the integral column:

$$\int 2x^{-3}e^{-x}\;dx$$

The Tabular Method - Remainders

$+$	x^{-1}	$-e^{-x}$	$-x^{-1}e^{-x}$
$-$	$-x^{-2}$	e^{-x}	$x^{-2}e^{-x}$
$+$	$2x^{-3}$		

The Tabular Method - Remainders

$+$	x^{-1}	$-e^{-x}$	$-x^{-1}e^{-x}$
$-$	$-x^{-2}$	e^{-x}	$x^{-2}e^{-x}$
$+$	$2x^{-3}$		

The final result is the sum of the entries in the last column, plus the remainder:

$$\int \frac{e^{-x}}{x} dx = -\frac{e^{-x}}{x} + \frac{e^{-x}}{x^2} + \int \frac{2e^{-x}}{x^3}\, dx$$