Areas Between Curves

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Then the area bounded:

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- *below* by the graph of $g(x)$
- on the *left* by the vertical line $x = a$
- on the *right* by the vertical line $x = b$

is given by:

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The assumption that $f(x) - g(x) \geq 0$ on $[a, b]$ *is necessary*.

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$$= \int_0^1 1\, dx = x]_0^1 = 1 - 0 = 1$$

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$$= (e^x + e^{-x})\Big]_0^1 = e + e^{-1} - 2$$

Question 1

Find the area between the curves

$$x \text{ and } x^3 \quad \text{between } x = 0 \text{ and } x = 1$$

1. 1/4
2. 1/2
3. 2
4. 1
5. 3
6. none of the above

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1. $A = 1/4$

Question 2

Find the area between the curves

$$\cos x \text{ and } \sin x \quad \text{between } x = 0 \text{ and } x = \frac{\pi}{4}$$

1. $\sqrt{2}$
2. $\sqrt{2}/2$
3. $\sqrt{2} - 1$
4. 1
5. $\sqrt{2} - 3$
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6. none of the above

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If neither $f(x) \geq g(x)$ nor $g(x) \geq f(x)$ on $[a, b]$, then the previous formula cannot be used for the entire interval.

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- $\cos x \leq \sin x$ when $\pi/4 \leq x \leq \pi/2$

We have to split

$$A = \int_0^{\pi/2} |\cos x - \sin x| \, dx$$

into two parts. The graphs cross at $x = \pi/4$ in this case.

Areas Between Curves

$$\int_0^{\pi/2} |\cos x - \sin x|\, dx$$

$$= \int_0^{\pi/4} [\cos x - \sin x] dx + \int_{\pi/4}^{\pi/2} [\sin x - \cos x] dx$$

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$$= (\sqrt{2} - 1) + (\sqrt{2} - 1) = 2\sqrt{2} - 2$$

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$$= (\sqrt{2} - 1) + (\sqrt{2} - 1) = 2\sqrt{2} - 2$$

The difficult part of this type of problem is usually finding the points where the curves of f and g cross.

Question 3

Find the area between the curves

$$x^3 \text{ and } x \quad \text{between } x = -1 \text{ and } x = 1$$

1. $1/3$
2. $1/4$
3. 0
4. 1
5. $1/2$
6. none of the above

Question 3

Find the area between the curves

$$x^3 \text{ and } x \quad \text{between } x = -1 \text{ and } x = 1$$

1.	$1/3$	4 .	1
2.	$1/4$	5.	$1/2$
3.	0	6.	none of the above

5. $A = 1/2$

Question 4

Find the area between the curves

$y = 2x + 1$ and $y = -x + 4$ between $x = 0$ and $x = 2$

1. $1/3$
2. $1/4$
3. 2
4. 3
5. $1/2$
6. none of the above

Question 4

Find the area between the curves

$y = 2x + 1$ and $y = -x + 4$ between $x = 0$ and $x = 2$

1. $1/3$
2. $1/4$
3. 2
4. 3
5. $1/2$
6. none of the above

4. $A = 3$ The graphs intersect at $x = 1$.

Question 5

Find the region bounded by the curves

$$y = 4x - x^2 \text{ and } y = x^2$$

1. $8/3$
2. $4/3$
3. $2/3$
4. 3
5. $1/8$
6. none of the above

Question 5

Find the region bounded by the curves

$$y = 4x - x^2 \text{ and } y = x^2$$

1. $8/3$
2. $4/3$
3. $2/3$
4. 3
5. $1/8$
6. none of the above

1. $A = 8/3$ The graphs intersect at $x = 0$ and $x = 2$.

Question 6

Find the region bounded by the curves

$$y = \sin(\pi x/2) \text{ and } y = x^2 - 2x$$

1.	$4/3$	4 .	$3/\pi$
2.	$\pi/3$	5.	$3\pi/8$
3.	$4/3 + 4/\pi$	6.	none of the above

Question 6

Find the region bounded by the curves

$$y = \sin(\pi x/2) \text{ and } y = x^2 - 2x$$

1.	$4/3$	4 .	$3/\pi$
2.	$\pi/3$	5.	$3\pi/8$
3.	$4/3 + 4/\pi$	6.	none of the above

3. $A = 4/3 + 4/\pi$ The graphs intersect at $x = 0$ and $x = 2$.