# Sections 5.1

Gene Quinn

# The Area Problem

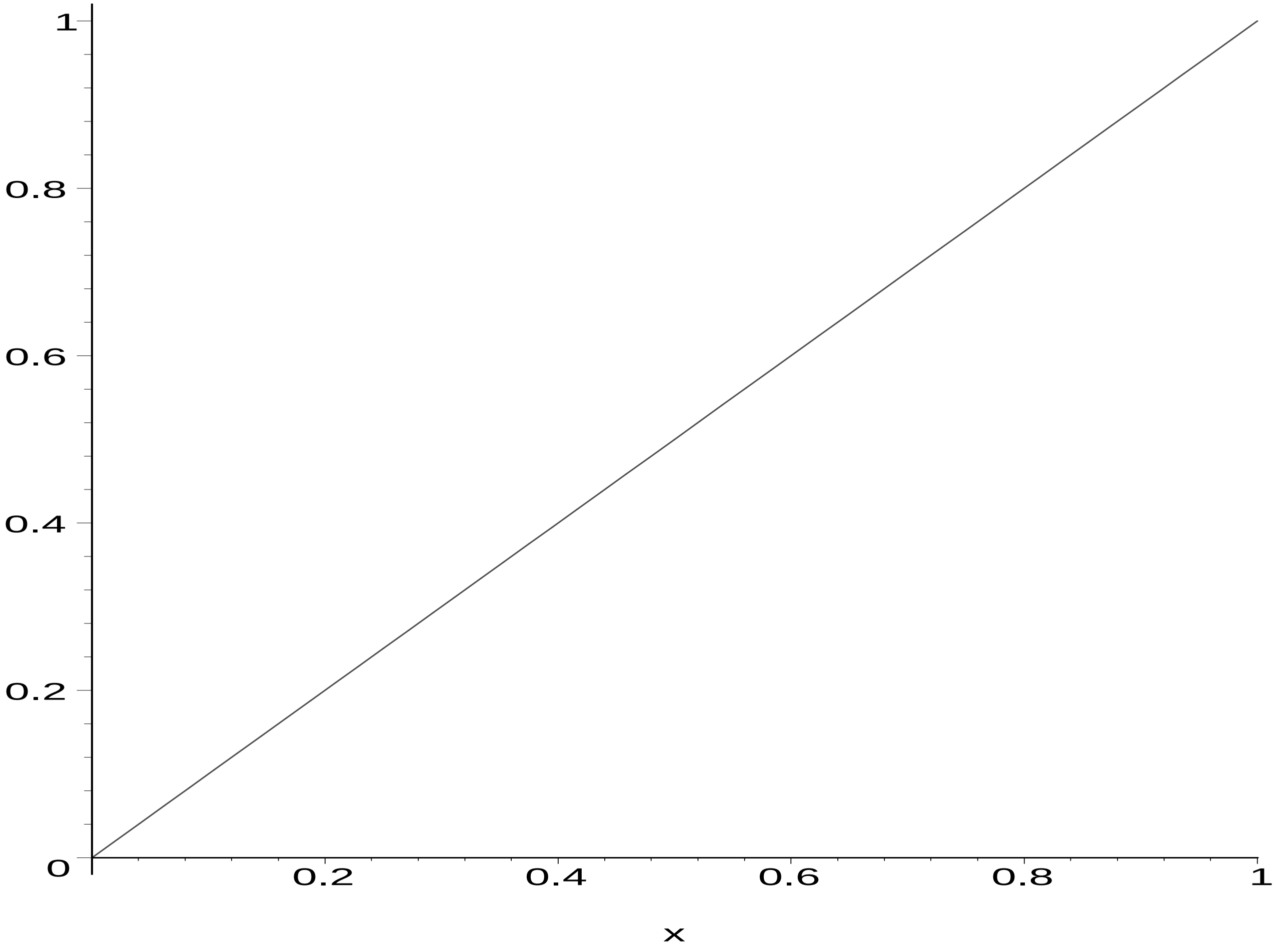
Suppose we want to find the area under the graph of the function

$$y = f(x) = x$$

between the $x$-coordinates $0$ and $1$.

# The Area Problem

Drawing a picture, we recognize the area as a right triangle.



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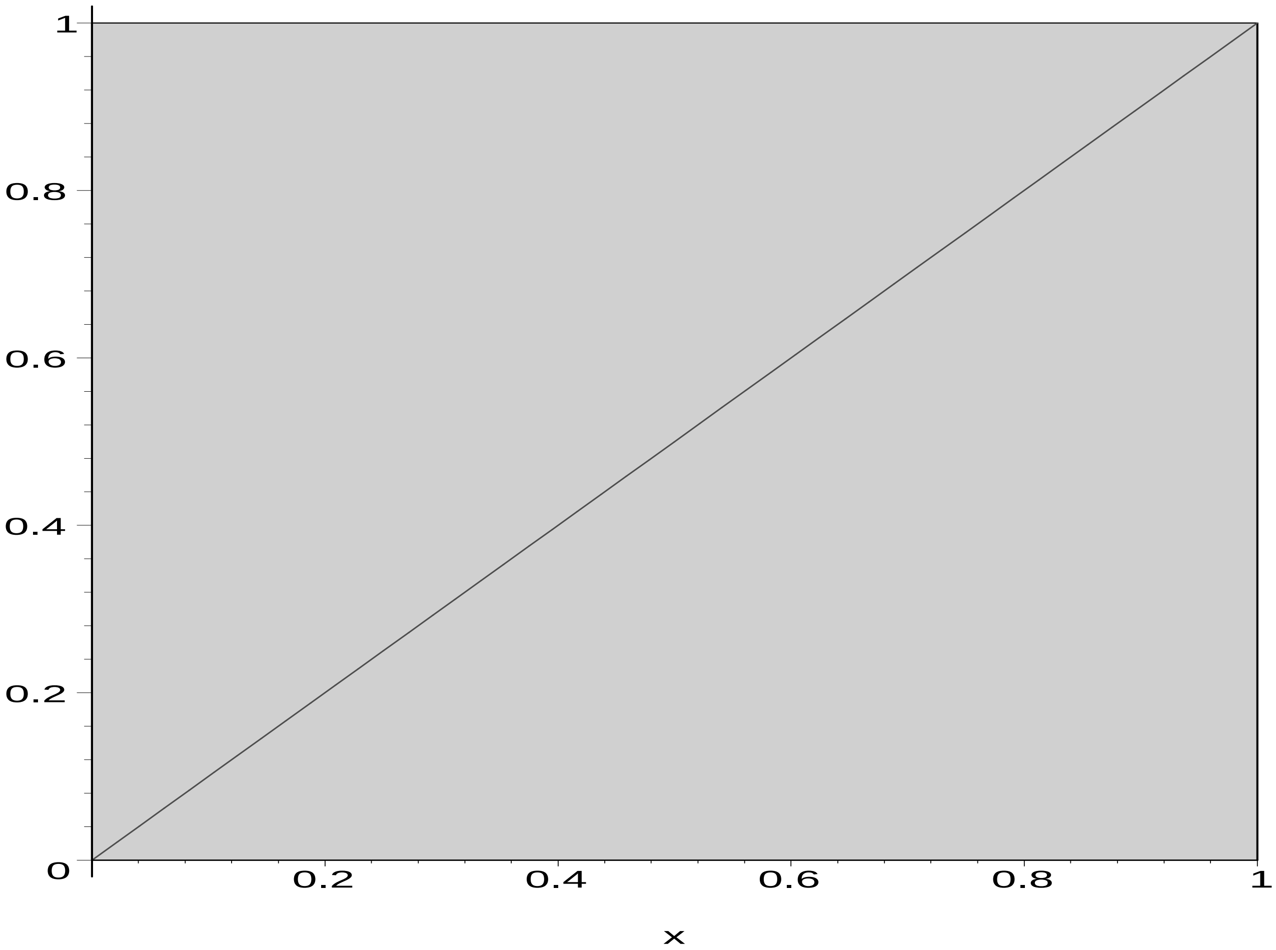
We would like to find a method of computing the area under the graph of a more general function.

One strategy is to approximate the area using a shape we know how to find the area of.

# The Area Problem

We can approximate the area under the graph by a rectangle with a corners at the origin and the point $(1, 1)$:



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The area of the rectangle is $1$, which overstates the area of the triangle, which we know is

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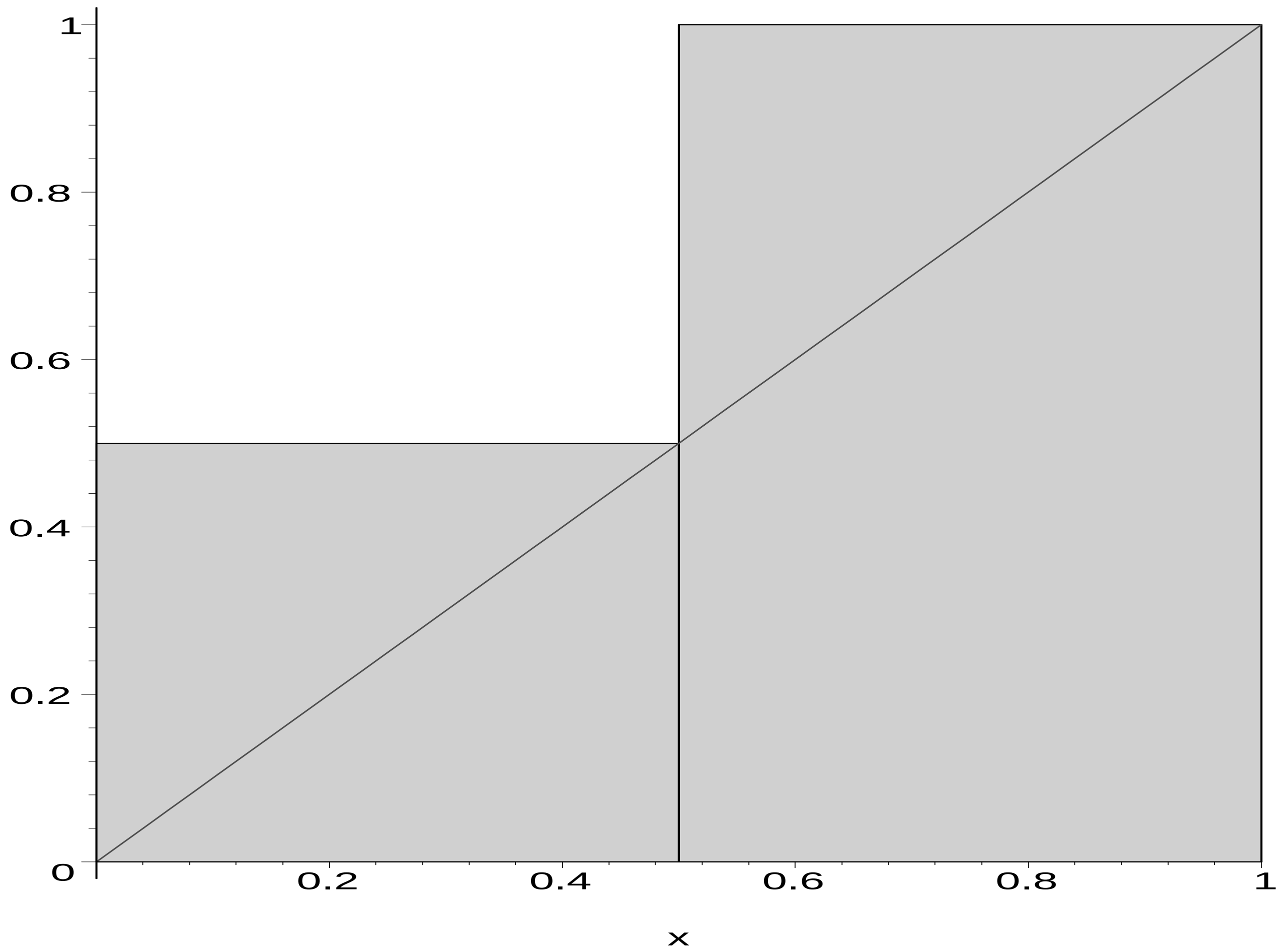
Suppose we divide the interval from $0$ to $1$ into two equal subintervals

$$\left[0, \frac{1}{2}\right] \quad \text{and} \quad \left[\frac{1}{2}, 1\right]$$

Now we can construct two rectangles, using the value of $f(x)$ at the right endpoint of each.

# The Area Problem

Now the picture looks like this:



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We improved the approximation by taking two rectangles, so now try four.

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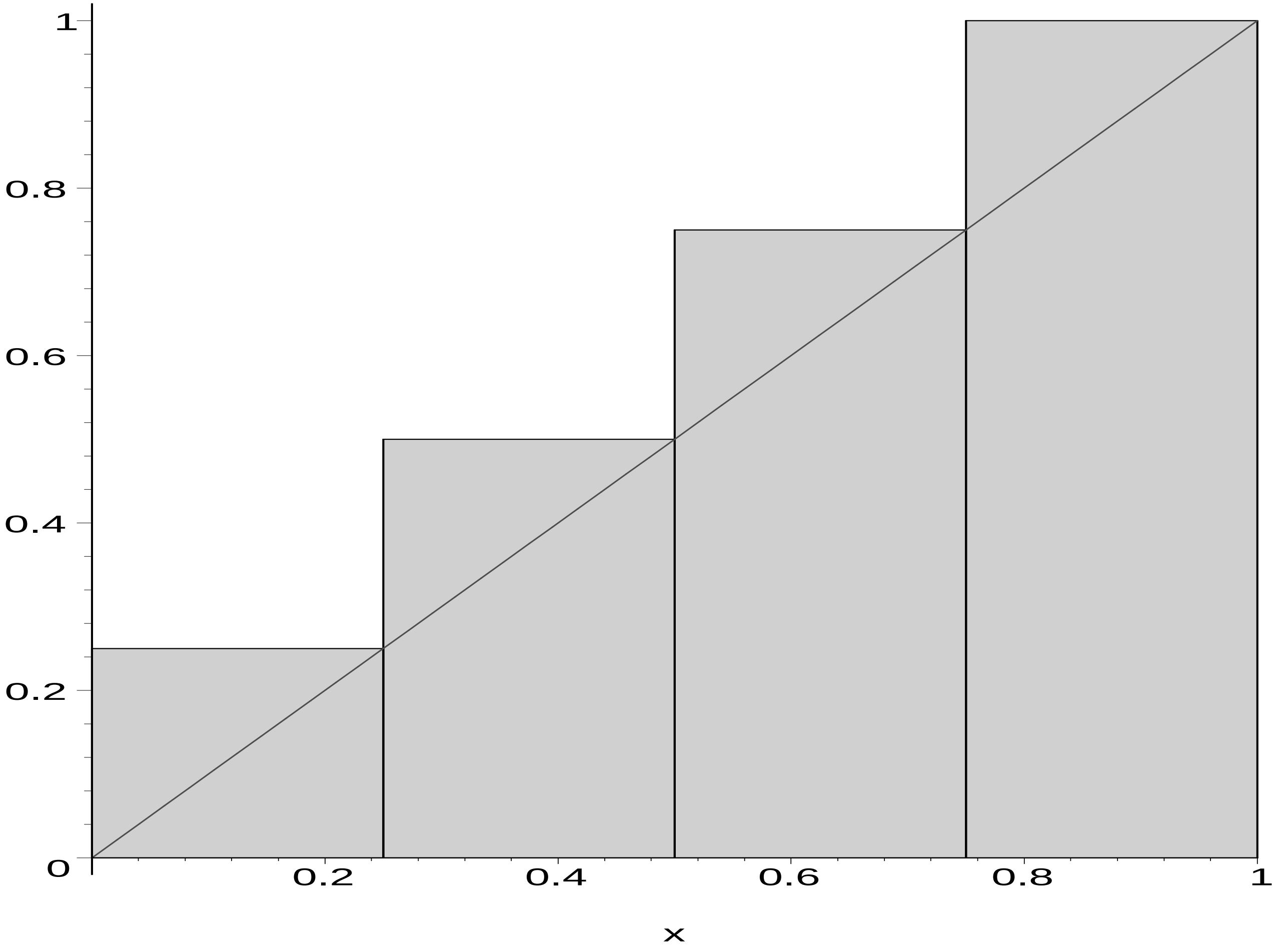

# The Area Problem

The combined area of the four rectangles is

$$R_4 = \frac{1}{4} \cdot f\left(\frac{1}{4}\right) + \frac{1}{4} \cdot f\left(\frac{2}{4}\right) + \frac{1}{4} \cdot f\left(\frac{3}{4}\right) + \frac{1}{4} \cdot f\left(\frac{4}{4}\right)$$

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We can write this expression as

$$R_4 = \frac{1}{4} \cdot \frac{1}{4} \cdot (1 + 2 + 3 + 4) \quad = \quad \frac{10}{16}$$

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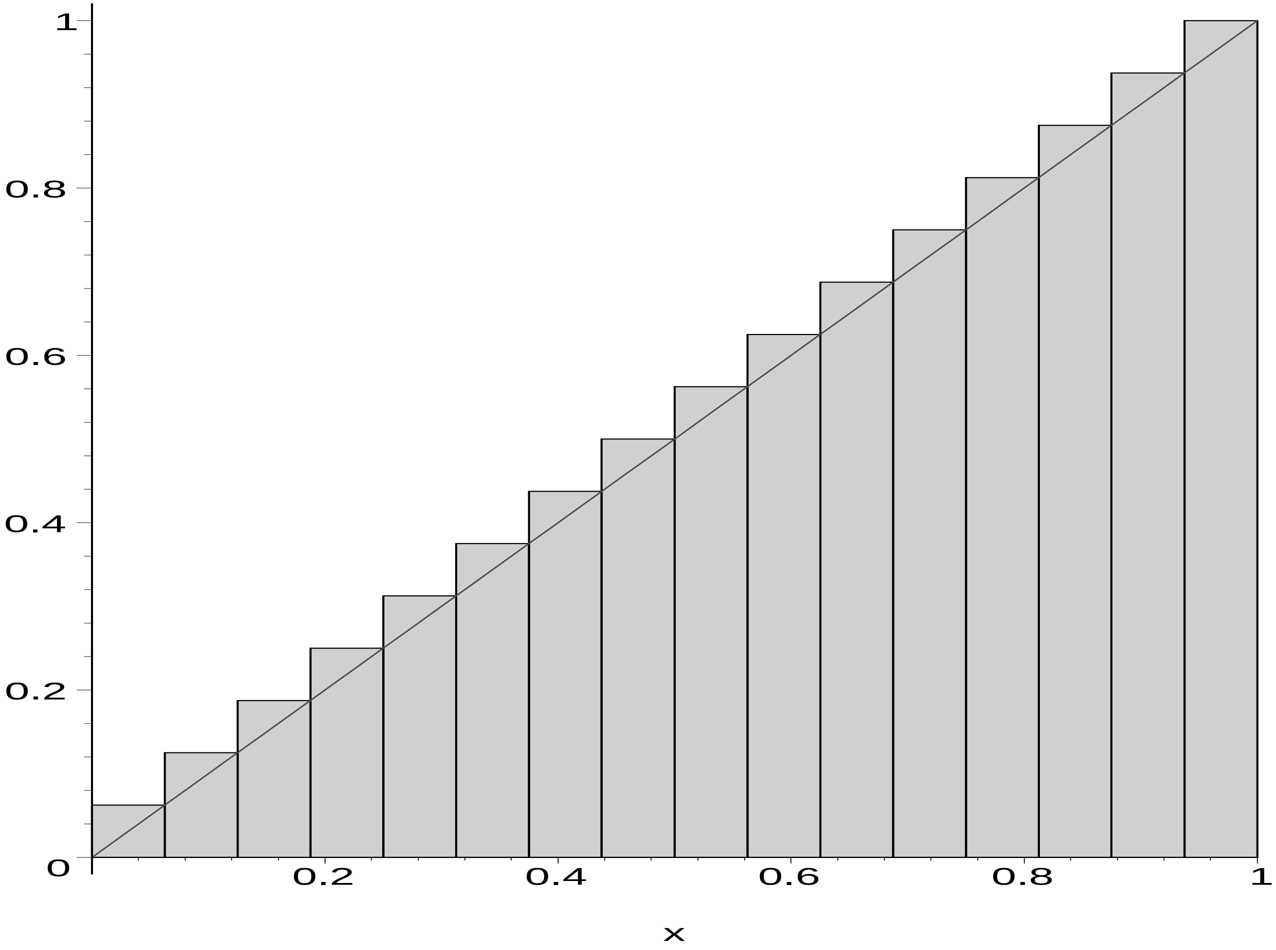
We can continue to, say, 16 rectangles.

# The Area Problem

The new picture is:



# The Area Problem

As before, when we write the expression for the total area of the 16 rectangles and collect terms, we get

$$R_{16} = \frac{1}{16} \cdot \frac{1}{16} \cdot (1 + 2 + \cdots + 15 + 16)$$

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$$1 + 2 + \cdots + n - 1 + n = \frac{n(n+1)}{2}$$

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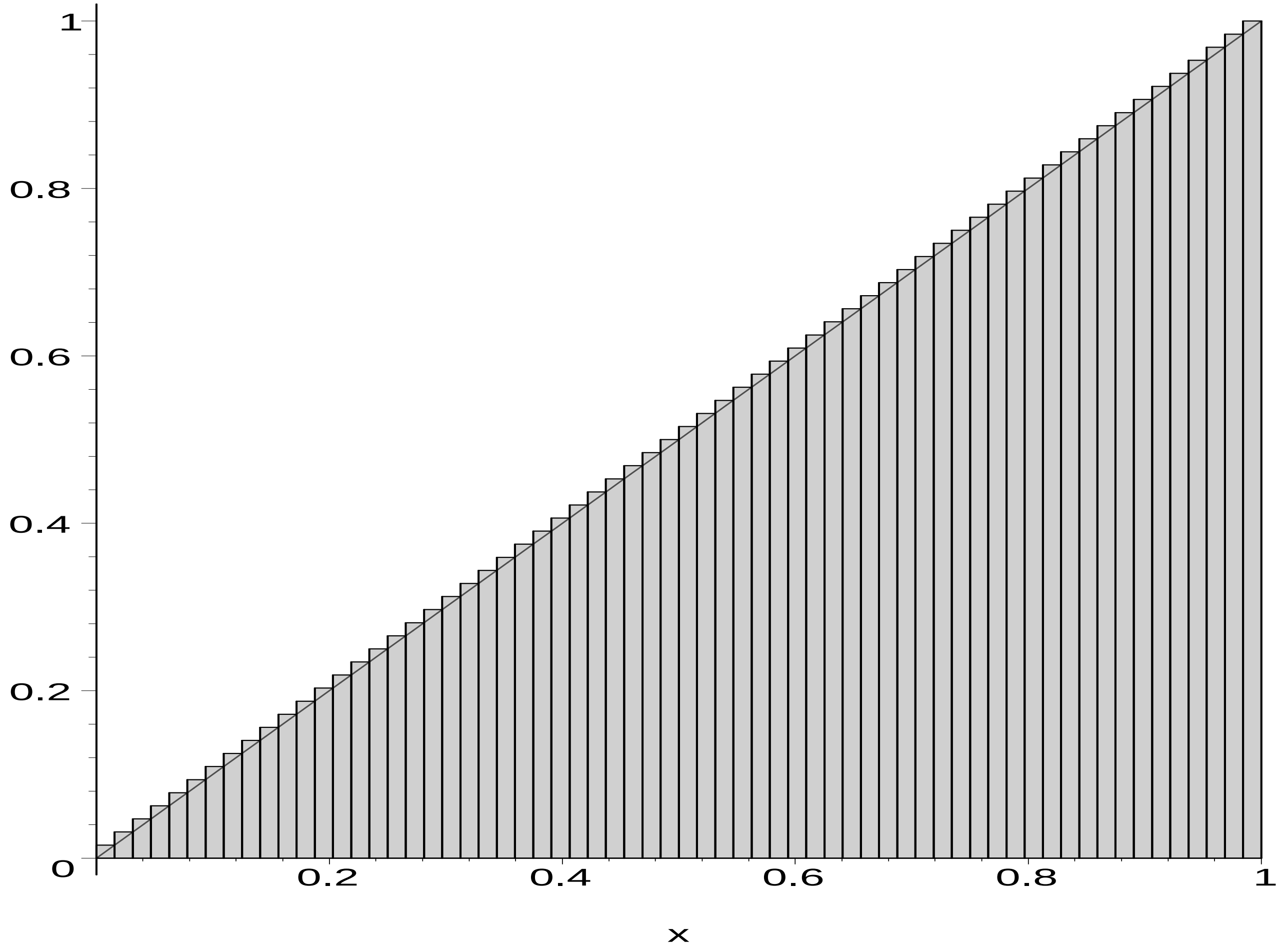
$$1 + 2 + \cdots + n - 1 + n = \frac{n(n+1)}{2}$$

Using this formula with $n = 16$,

$$R_{16} = \frac{1}{16} \cdot \frac{1}{16} \cdot \frac{16 \cdot 17}{2} = \frac{136}{256} = .531$$

# The Area Problem

With 64 rectangles, the picture is:



# The Area Problem

With 64 rectangles, the area is

$$R_{64} = \frac{1}{64} \cdot \frac{1}{64} \cdot \frac{64 \cdot 65}{2} = \frac{2080}{4096} = .508$$

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In principle there is no limit to the number of rectangles we can have, and apparently the approximation improves as we take more.

# The Area Problem

In the general case, say $n$ rectangles, their combined area is

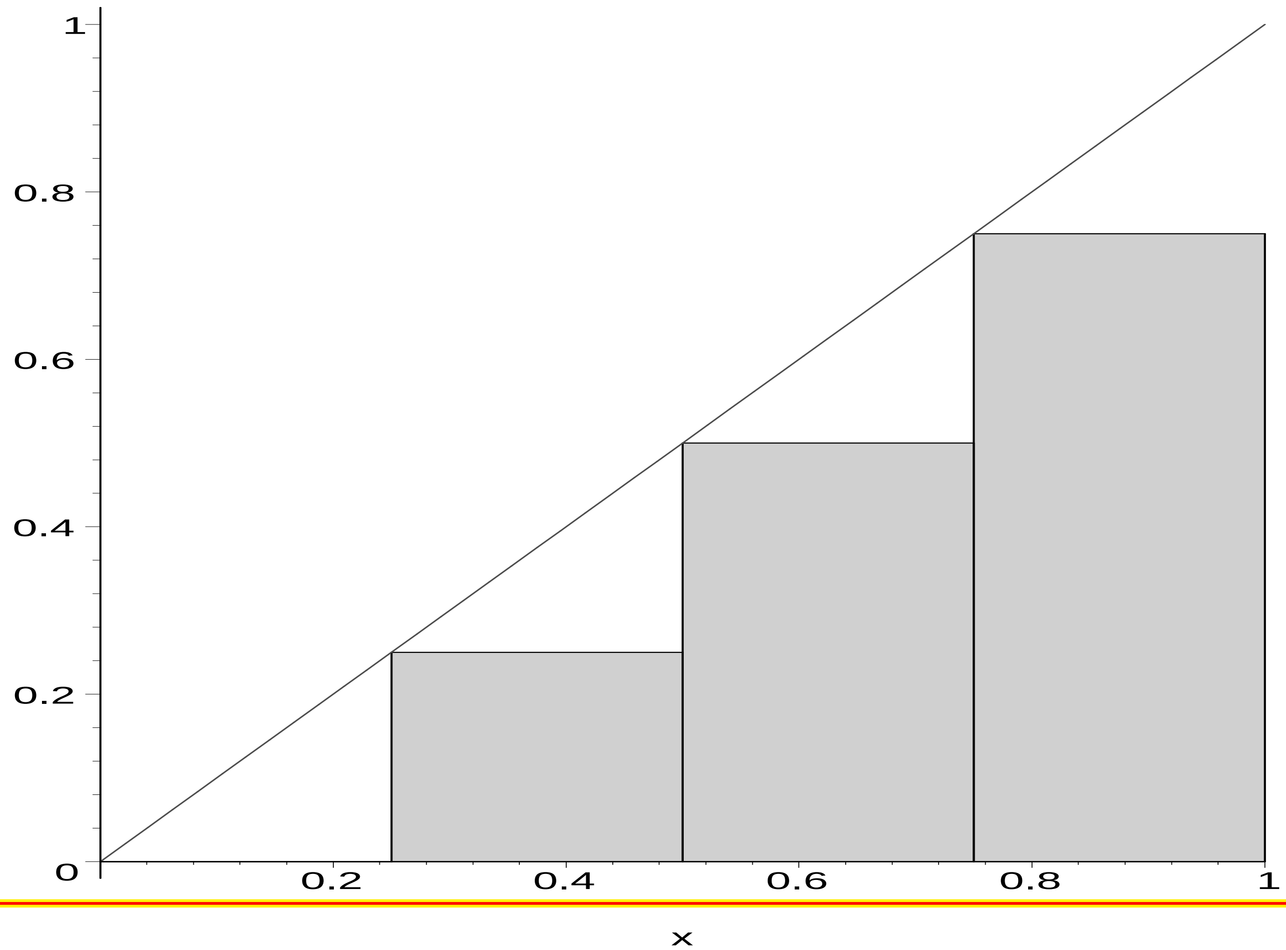
$$R_n = \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{n \cdot (n+1)}{2}$$

$$= \frac{n+1}{2n}$$

$$= \frac{1}{2} + \frac{1}{2n}$$

# The Area Problem

We could have chosen function value at the left endpoint of each interval, which for four rectangles produces the following picture:



# The Area Problem

The combined area of the four rectangles using left endpoints is

$$L_4 = \frac{1}{4} \cdot f\left(\frac{0}{4}\right) + \frac{1}{4} \cdot f\left(\frac{1}{4}\right) + \frac{1}{4} \cdot f\left(\frac{2}{4}\right) + \frac{1}{4} \cdot f\left(\frac{3}{4}\right)$$

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We can write this expression as

$$L_4 = \frac{1}{4} \cdot \frac{1}{4} \cdot (0 + 1 + 2 + 3) \quad = \quad \frac{6}{16}$$

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For rectangles whose height is the function value at the left endpoint, the only change is in the summation.

Instead of summing the integers from $1$ to $n$, we are summing the integers from $0$ to $n - 1$.

# The Area Problem

Of course, the sum of the integers from $0$ to $n-1$

is the same as the sum of the integers from $1$ to $n-1$, which we know is

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In the general case of $n$ rectangles with the height equal to the function value at the **left** endpoint, the combined area is

$$L_n = \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{n \cdot (n-1)}{2}$$

$$= \frac{n-1}{2n}$$

$$= \frac{1}{2} - \frac{1}{2n}$$

# The Area Problem

If we call the area below the graph $A$, we can write the following inequality:

$$L_n \quad = \quad \frac{1}{2} - \frac{1}{2n} \quad \leq \quad A \quad \leq \quad \frac{1}{2} + \frac{1}{2n} \quad = \quad R_n$$

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$$L_n = \frac{1}{2} - \frac{1}{2n} \leq A \leq \frac{1}{2} + \frac{1}{2n} = R_n$$

Now take limits as the number of rectangles increases without bound, that is, as $n \to \infty$

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \left( \frac{1}{2} - \frac{1}{2n} \right) \leq \lim_{n\to\infty} A$$

$$\leq \lim_{n\to\infty} \left( \frac{1}{2} + \frac{1}{2n} \right) = \lim_{n\to\infty} R_n$$

# The Area Problem

The center term is just a constant and by the squeeze theorem the area $A$ must be $1/2$:

$$\frac{1}{2} \quad \leq \quad A \quad \leq \frac{1}{2}$$

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Now we would like to write formulas for $R_n$ and $L_n$:

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The width of each rectangle will be

$$\Delta x = \frac{b - a}{n}$$

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The **right** endpoint of the $i^{th}$ rectangle is

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So, with this definition of $x_i$,

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

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Similarly, the **left** endpoint of the $i^{th}$ rectangle is

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$$x_i = a + (i - 1) \cdot \Delta x$$

So with $x_i$ as defined above,

$$L_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

# Definition of Area

Now we state the definition of the area under the graph of a continuous function $f$:

**Definition:** The **area** $A$ of the region $S$ that lies under the graph of a continuous function $f$ is the limit of the sum of the areas of approximating rectangles,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

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It can be shown that this limit always exists if $f$ is continuous.

It can also be shown that we get the same value if we use $L_n$ instead of $R_n$.

# Definition of Area

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

In fact, we get the same value if we choose $x_i$ to be **any** value $x_i^*$ in the $i^{th}$ interval.