MTH125 Stewart Section 1.6

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One-to-One Functions

A function is said to be *one-to-one* if it never takes the same value twice; That is, for every x_1, x_2 in the domain of f,

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The function

$$f(x) = x^2$$

is not one-to-one because

$$f(-2) = f(2) \quad \text{but} \quad -2 \neq 2$$

Inverse of a one-to-one Function

Suppose f is a one-to-one function with domain A and range B. The its *inverse function* f^{-1} has domain B and range A and is defined by:

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Composing f with f^{-1} we have the so-called *cancellation equations*

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \text{ for every } x \in A$$
$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x \text{ for every } x \in B$$

Finding the Inverse of a 1:1 Function

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3) Interchange y and x, that is, replace all x's with y's and all y's with x's.

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Step 2: solve the equation for y in terms of x.

$$\begin{aligned} y &= 3x + 4 \\ y - 4 &= 3x \\ (y - 4)/3 &= x \end{aligned}$$

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Step 3: interchange x and y:

$$\frac{y-4}{3} = x$$

becomes

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Check:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = \frac{(3x-4)+4}{3} = \frac{3x}{3} = x$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = 3\left(\frac{x-4}{3}\right) + 4 = x - 4 + 4 = x$$

Logarithmic Functions

Note that the exponential function with base a

$$y = f(x) = a^x, \quad a \neq 1$$

passes the horizontal line test, so it's 1:1 and therefore has an inverse function f^{-1}.

We define the logarithmic function $\log_a x$ by

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Using the cancellation equations, we have

$$\log_a(a^x) = x \quad \forall x \in R$$

and

$$a^{\log_a x} = x \quad \forall x > 0$$

Logarithmic Functions

If x and y are positive numbers, then

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a(\frac{x}{y}) = \log_a x - \log_a y$$

$$\log_a(x^r) = r \log_a x$$

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The logarithm with base e is called the *natural logarithm* and has the following special notation:

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$$\ln x = y \Leftrightarrow e^y = x$$

Using the cancellation equations, we have

$$\ln(e^x) = x \quad \forall x \in R$$

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Sample Problem with Logarithms

Write the expression

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Now apply the first property:

$$\log_a x + \log_a \sqrt{y} = \log_a(x\sqrt{y})$$