The Uniform Distribution

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If the experiment is dealing a 13-card bridge hand, each of the ${}_{52}C_{13}$ possible hands is considered to be equally likely.

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An analogous model with a continuous sample space would be the following experiment:

Choose a real number at random from the interval $[0, 1]$, with each number in the interval equally likely to be chosen.

Probability Density Function

If X is a random variable with the uniform distribution, its probability density function (pdf) is:

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A simpler way of writing this is

$$f_X(x) = 1, \quad 0 \le x \le 1$$

with the implicit understanding that outside of its support (the closed interval $[0, 1]$), the pdf is zero.

It is easy to verify that, if we integrate the pdf over its support, the result is 1:

$$\int_0^1 1\, dx = 1$$

Cumulative Distribution Function

The cumulative distribution function for the uniform distribution is

$$F_X(x) \;=\; P(X \leq x) \;=\; \int_0^x 1\,dt \;=\; x$$

Moments

The expected value of a uniform random variable is

$$\mathsf{E}(X) \;=\; \int_0^1 x \cdot 1 \, dx \;=\; \frac{1}{2}$$

The expected value of its square is

$$\mathsf{E}(X^2) \;=\; \int_0^1 x^2 \cdot 1 \, dx \;=\; \frac{1}{3}$$

Its variance is

$$\mathsf{Var}(X) \;=\; \mathsf{E}(X^2) - [\mathsf{E}(X)]^2 \;=\; \frac{1}{3} - \frac{1}{4} \;=\; \frac{1}{12}$$

Role in Computer Simulations

The uniform distribution plays an important role in computer simulations or Monte Carlo experiments.

Most software packages, calculators, and spreadsheets have a built-in function, often called $RAND()$, that returns a uniform random variate.

One technique for producing a random variate with cumulative distribution $F(x)$ is the following:

- 1) generate a uniform random variate X
- 2) The variable $Y = F^{-1}(X)$ will have the desired distribution

Congruential Generators

A very common technique for generating psuedorandom numbers with a uniform distribution is based on a sequence of nonnegative integers $\{x_n\}$ defined by a recursive formula, say

$$x_{n+1} \; = \; k \cdot x_n \, (\text{modulo}(m)) \, , \quad n = 0, 1, \ldots$$

A result from number theory states that it is possible to choose k and m so that regardless of the starting value or *seed* x_0, every number in the set $\{0, 1, \ldots, m-1\}$ will appear before any of them repeats.

If m is taken to be a large number, say $2^{31} - 1$, each of the roughly 2 billion integers in the sequence will appear once before any repeat.

Congruential Generators

If we generate a recursive sequence

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and divide each entry by m, we obtain a sequence of numbers in the interval $[0, 1]$ that *may* be a reasonable approximation to the outcome of the following experiment:

Choose a number from the interval $[0, 1]$ at random, with each number being equally likely to be chosen. Repeat this procedure independently m times.

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Choose a number from the interval $[0, 1]$ at random, with each number being equally likely to be chosen. Repeat this procedure independently m times.

We call this a *psuedorandom* number generator because it is completely deterministic. While there will be no repeats in the first m integers in the recursive sequence, once the first repeat occurs, the next m terms exactly reproduce the sequence of the first m terms. Also, given the same seed x_0, the sequence produced will always be the same.