1) Random variable Y has density function:

$$f(y) = \frac{1}{2}y^2e^{-y}, \quad y \in [0, \infty)$$

Find the density function of the random variable $U = Y^2$.

Solution: Let

$$u = h(y) = y^2 \quad \text{then} \quad y = h^{-1}(u) = \sqrt{u}, \quad u \in [0, \infty)$$

and

$$\frac{d}{du}h^{-1}(u) = \frac{1}{2\sqrt{u}}$$

so

$$f_u(u) = f_y(h^{-1}(u)) \cdot \left|\frac{d}{du}h^{-1}(u)\right|$$

and on substitution

$$f_u(u) = \frac{1}{2}(\sqrt{u})^2e^{-\sqrt{u}} \cdot \frac{1}{2\sqrt{u}} = \frac{\sqrt{u}}{4}e^{-\sqrt{u}}$$

2) A random vector $Y = \{Y_1, Y_2, \ldots, Y_n\}$ has n independent, identically distributed components each with density function

$$f(y_i) = \frac{1}{\beta}e^{-y/\beta}, \quad y \in [0, \infty)$$

a) Find the density function of $U = Y_1 + Y_2 + \cdots + Y_n$.

Solution: Observe that each Y_i has an exponential distribution, with moment-generating function

$$m_y(t) = \frac{1}{1 - \beta t}$$

The moment-generating function of the sum of independent random variables is the product of their moment-generating functions, so by Theorem 6.2,

$$m_u(t) = \Pi_{i=1}^n m_{y_i}(t) = \Pi_{i=1}^n \frac{1}{1 - \beta t} \quad = \quad (1 - \beta t)^{-n}$$

This function can be recognized as the moment-generating function of a gamma random variable with parameters $\alpha = n$ and $\beta = \beta$, so

$$U \sim \text{ gamma}(n, \beta)$$

b) Find the mean and variance of U.

Since U is a gamma random variable, the mean μ and variance σ^2 of U are given by:

$$\mu = \alpha\beta = n\beta \quad \text{and} \quad \sigma^2 = \alpha\beta^2 = n\beta^2$$

3) A random variable Y has density function

$$\begin{cases} f(y) & \text{if} \quad 0 \le y \le 1 \\ 0 & \quad\; otherwise \end{cases}$$

Find the density function and interval of support for the random variable

$$U = aY, \quad a > 0$$

Solution: Let

$$u = h(y) = ay \quad \text{then} \quad y = h^{-1}(u) = \frac{1}{a} \cdot u, \quad u \in [0, a]$$

and

$$\frac{d}{du} h^{-1}(u) = \frac{1}{a}$$

so

$$f_u(u) = f_y(h^{-1}(u)) \cdot \left| \frac{d}{du} h^{-1}(u) \right|$$

and on substitution

$$f_u(u) = \frac{1}{a} f(u/a)$$

4) A random vector $Y = \{Y_1, Y_2, \ldots, Y_n\}$ has n independent, identically distributed components each with a uniform distribution on $[0, 1]$.

The a) Find the density function $g_{(k)}(y_k)$ of the k^{th} order statistic, where $1 \le k \le n$. (Section 6.7)

By Theorem 6.5, the density function of the k^{th} order statistic ($1 \le k \le n$) is:

$$g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!} [F(y)]^{k-1} [1 - F(y)]^{n-k} f(y_k), \quad 0 \le y \le 1$$

The density function $f(y)$ and cumulative distribution function $F(y)$ are:

$$f(y) = 1 \quad \text{and} \quad F(y) = \int_0^y f(t)\, dt = \int_0^y dt = y, \quad 0 \le y \le 1$$

so

$$g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!}[y]^{k-1}[1-y]^{n-k} \cdot 1, \quad 0 \le y \le 1$$

If k is a positive integer,

$$\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}\,dt = (k-1)!$$

so letting $\alpha = k$ and $\beta = n - k + 1$, we can write the density function as:

$$g_{(k)}(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}[y]^{\alpha-1}[1-y]^{\beta-1}, \quad 0 \le y \le 1$$

which is a beta density.

b) Find the mean and variance of the k^{th} order statistic $Y_{(k)}$. (hint: see if you can recognize the density function in part a) as one that appears in the back cover of the text).

Solution: From the back cover of the text, the mean is:

$$\mu = \frac{\alpha}{\alpha+\beta} = \frac{k}{n-1}$$

and the variance is:

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{k(n-1)}{n \cdot (n-1)^2}$$

5) Suppose Y_1 has a chi-sqare distribution with 6 degrees of freedom, and Y_2 has a chi-square distribution with 15 degrees of freedom, and Y_1 and Y_2 are independently distributed.

a) Identify the distribution of the random variable $U = Y_1 + Y_2$.

Solution: Since Y_1 and Y_2 are independent, the moment-generating function of their sum will be the product of their moment-generating functions (using the back cover of the text),

$$m_u(t) = (1 - 2t)^{-6/2} (1 - 2t)^{-15/2} = (1 - 2t)^{-21/2}$$

which is the moment-generating function of a chi-square distribution with $6 + 15 = 21$ degrees of freedom.

b) What is the density function of U?

From the back cover of the text with $\nu = 21$,

$$f(y) = \frac{(y)^{(21/2)-1} e^{-y/2}}{2^{(21/2)} \Gamma(21/2)} \quad y > 0$$

c) What are the mean and variance of U?

Again from the back cover of the text,

$$\mu = \nu = 21 \quad \text{and} \quad \sigma^2 = 2\nu = 42$$