Name:

1) Time to respond for 911 calls was measured for 75 calls in two cities, $C1$ and $C2$. The results of the study were:

Measure	C1	C2
Sample size	75	75
Sample mean	5.4	6.1
Sample variance	1.57	2.54

This problem is solved exactly like Example 8.3 in the text. **a)** Estimate the difference in the mean response time for the two cities.

The point estimate of $\mu_1 - \mu_2$ is

$$\overline{y}_1 - \overline{y}_2 = 5.4 - 6.1 = -0.7$$

b) Find a bound for the error of estimation. See Example 8.3 in the text. The standard deviation of the estimated difference is:

$$\sigma_{\overline{Y}_1 - \overline{Y}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \approx \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{1.57}{75} + \frac{2.54}{75}} = 0.234$$

The probability that the error of estimation is less than $\sigma_{\overline{Y}_1 - \overline{Y}_2} \approx 2(0.234) = 0.468$ is .95. (two standard deviations)

2) An exit poll of $1,000$ voters finds that 530 supported a certain candidate.

a) Estimate the proportion of the voting population that supports the candidate.

The estimator is

$$\hat{p} = \frac{y}{n} = \frac{530}{1000} = 0.530$$

b) Find a bound for the error of estimation. The standard error for the estimator $\hat{p}$ is given by:

$$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.53 \cdot 0.47}{1000}} = 0.0158$$

With probability .95, the error of estimation is less than $2\sigma_{\hat{p}} = 0.0316$

3) Suppose Y is a single observation from an exponential distribution with unknown mean θ.

a) Use the method of moment-generating functions to show that $2Y/\theta$ is a pivotal quantity having a χ^2 distribution with two degrees of freedom.

From the table in the back of the text, the moment-generating function of y is

$$m_y(t) = (1 - \theta t)^{-1}$$

so the moment-generating function of $u = 2Y/\theta$ is

$$m_u(t) = m_y\left(\frac{2}{\theta}t\right) = (1 - 2t)^{-1}$$

Comparing this to the table in the back cover of the text, we see this is the moment-generating function of a Chi-square random variable with 2 degrees of freedom.

b) Use the pivotal quantity from part a) to derive a 90% confidence interval for θ. Compare your result with Example 8.4 in the text.

We need to find bounds L and U so that

$$P(L \leq \chi^2_2 \leq U) = 0.90$$

There are many intervals that satisfy this inequality, but only one is symmetric in the sense that the areas under the graph of the density function from zero to L and from U to infinity are both 0.05.

We can find the value of L and U from the table in the back of the text (use the row with df=2). The value of L is under the column heading $\chi^2_{0.950}$ and is 0.102587. The value of U is under the column heading $\chi^2_{0.050}$ and is 5.99147. You can also obtain these from a spreadsheet with the formulas

$$L = \text{CHIINV(0.95,2)} \quad \text{and} \quad U = \text{CHIINV(0.05,2)}$$

so, for a Chi-square variable with 2 degrees of freedom,

$$P(0.102587 \leq \chi^2_2 \leq 5.99147) = 0.90$$

substituting $2Y/\theta$ into this inequality for χ^2_2 we get:

$$P\left(0.102587 \leq \frac{2Y}{\theta} \leq 5.99147\right) = 0.90$$

which rearranges to

$$P\left(\frac{2Y}{5.99147} \leq \theta \leq \frac{2Y}{0.102587}\right) = 0.90$$

Note that this produces the same confidence interval as the one obtained in Example 8.4.

4) Now suppose $(Y_1, \ldots, Y_9)$ is a sample of size $n = 9$ from an exponential distribution with mean θ.

a) Use the method of moment-generating functions to show that

$$\frac{2}{\theta}\sum_{i=1}^{9} Y_i$$

is a pivotal quantity having a χ^2 distribution with 18 degrees of freedom.

Because the Y_is are independent, moment-generating function of their sum is the product of their individual moment-generating functions, so

$$m_{\sum Y_i}(t) = \prod_{i=1}^{9} m_y(t) = (1 - \theta t)^{-9}$$

and the moment-generating function of $2/\theta$ times the sum of the Y_is is:

$$m_{2/\theta \sum}(t) = m_{\sum Y_i}(2t/\theta) = (1 - 2t)^{-9}$$

which from the tables in the back of the text is the moment-generating function of a Chi-square variable with 18 degrees of freedom.

b) Use the pivotal quantity from part a) to derive a 95% confidence interval for θ.

As before we start by finding L and U such that

$$P(L \leq \chi^2_{18} \leq U) = 0.95$$

Either from the table in the text, or using

$L = \text{CHIINV(0.975,18)} = 8.23$ and $U = \text{CHIINV(0.025,18)} = 31.53$

we write

$$P\left(8.23 \leq \frac{2\sum_{i=1}^{9} Y_i}{\theta} \leq 31.53\right) = 0.95$$

which rearranges to

$$P\left(\frac{2\sum_{i=1}^{9} Y_i}{31.53} \le \theta \le \frac{2\sum_{i=1}^{9} Y_i}{8.23}\right) = 0.95$$

5) Let $(Y_1, \ldots, Y_5)$ be a sample of size $n = 5$ from a gamma distribution with $\alpha = 2$ and β unknown.

a) Use the method of moment-generating functions to show that

$$\frac{2}{\beta}\sum_{i=1}^{5} Y_i$$

is a pivotal quantity having a χ^2 distribution with 20 degrees of freedom.

Because the Y_is are independent, moment-generating function of their sum is the product of their individual moment-generating functions, so

$$m_{\sum Y_i}(t) = \prod_{i=1}^{5} m_y(t) = (1 - \beta t)^{-2}$$

and the moment-generating function of $2/\beta$ times the sum of the Y_is is:

$$m_{2/\beta \sum}(t) = m_{\sum Y_i}(2t/\beta) = (1 - 2t)^{-10}$$

which from the tables in the back of the text is the moment-generating function of a Chi-square variable with 20 degrees of freedom.

b) Use the pivotal quantity from part a) to derive a 95% confidence interval for θ.

As before we start by finding L and U such that

$$P(L \le \chi^2_{20} \le U) = 0.95$$

Either from the table in the text, or using

$L = \text{CHIINV}(0.975{,}20) = 9.59$ and $U = \text{CHIINV}(0.025{,}20) = 34.17$

we write

$$P\left(9.59 \le \frac{2\sum_{i=1}^{5} Y_i}{\beta} \le 34.17\right) = 0.95$$

which rearranges to

$$P\left(\frac{2\sum_{i=1}^{5} Y_i}{34.17} \leq \beta \leq \frac{2\sum_{i=1}^{5} Y_i}{9.59}\right) = 0.95$$