1) Random variable X and Y have joint density function:

$$f(x,y) = \begin{cases} a \cdot (x^2y + 2xy^2) & \text{if} \quad 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

a) Find the value of a that makes f a valid joint density function.

$$\int_0^1 \int_0^2 a \cdot (x^2y + 2xy^2)\, dy\, dx = \frac{10a}{3} \quad \text{so} \quad a = \frac{3}{10}$$

b) Find the joint cumulative distribution function $F(x,y)$

$$F(x,y) = \int_0^x \int_0^y f(u,v)\, dv\, du = \frac{x^3y^2 + 2x^2y^3}{20}$$

c) Find marginal density function of x, $f_x(x)$

$$f_x(x) = \int_0^2 f(x,y)\, dy = \frac{8}{5}x + \frac{3}{5}x^2$$

d) Find marginal density function of y, $f_y(y)$

$$f_y(y) = \int_0^1 f(x,y)\, dx = \frac{1}{10}y + \frac{3}{10}y^2$$

d) Find the expected value and variance of X, $E(X)$ and $V(X)$

$$E(x) = \int_0^1 \int_0^2 x \cdot f(x,y)\, dy\, dx = \int_0^1 x \cdot f_x(x)\, dx = \frac{41}{60}$$

$$E(x^2) = \int_0^1 \int_0^2 x^2 \cdot f(x,y)\, dy\, dx = \int_0^1 x^2 \cdot f_x(x)\, dx = \frac{13}{25}$$

$$V(x) = E(x^2) - [E(x)]^2 = \frac{13}{25} - \left(\frac{41}{60}\right)^2 = \frac{191}{3600}$$

Alternatively, one could use the definition directly and evaluate

$$V(x) = \int_0^1 \int_0^2 \left(x - \frac{41}{60}\right)^2 f(x,y)\, dy\, dx$$

The result would be the same.

e) Find the expected value and variance of Y, $E(Y)$ and $V(Y)$

$$E(y) = \int_0^1 \int_0^2 y \cdot f(x,y)\, dy\, dx = \int_0^1 y \cdot f_y(y)\, dy = \frac{22}{15}$$

$$E(y^2) = \int_0^1 \int_0^2 y^2 \cdot f(x,y)\, dy\, dx = \int_0^1 y^2 \cdot f_y(y)\, dy = \frac{58}{25}$$

$$V(y) = E(y^2) - [E(y)]^2 = \frac{58}{25} - \left(\frac{22}{15}\right)^2 = \frac{38}{225}$$

Alternatively, one could use the definition directly and evaluate

$$V(y) = \int_0^1 \int_0^2 \left(y - \frac{22}{15}\right)^2 f(x,y)\, dy\, dx$$

f) Find the covariance of X and Y, $\text{Cov}(X,Y)$

$$E(xy) = \int_0^1 \int_0^2 xy \cdot f(x,y)\, dy\, dx = 1$$

Then

$$Cov(x,y) = E(xy) - E(x)E(y) = 1 - \frac{41}{60} \cdot 2215 = \frac{-1}{450}$$

Alternatively, one could evaluate

$$\int_0^1 \int_0^2 \left(x - \frac{41}{60}\right)\left(y - \frac{22}{15}\right) \cdot f(x,y)\, dy\, dx$$

with the same result.

g) Find the conditional density $f_{x|y}(x)$ of X given Y The conditional density is:

$$f_{x|y} = \frac{f(x,y)}{f_y(y)} = \frac{\frac{3x^2y+6xy^2}{10}}{\frac{y+3y^2}{10}}$$

where y is taken to be a constant equal to the given value.

h) Find the conditional density $f_{y|x}(y)$ of Y given X The conditional density is:

$$f_{y|x} = \frac{f(x,y)}{f_x(x)} = \frac{\frac{3x^2y+6xy^2}{10}}{\frac{3x^2+8x}{5}}$$

where x is taken to be a constant equal to the given value.

2) A random variable Y has density function

$$f(y) = \begin{cases} a \cdot e^{-(2x+y)} & \text{if} \quad x, y \in [0, \infty) \\ 0 & \quad\quad \text{elsewhere} \end{cases}$$

a) Find the value of a that makes f a valid joint density function.

$$\int_0^\infty \int_0^\infty a \cdot e^{-(2x+y)}\, dy\, dx = \quad \text{so} \quad a = 2$$

b) Find the joint cumulative distribution function $F(x, y)$

$$F(x,y) = \int_0^x \int_0^y 2 * e^{(-2u-v)} dv\, du = 1 - e^{-y} - e^{-2x} + e^{-(2x+y)}$$

c) Find marginal density function of x, $f_x(x)$

$$f_x(x) = \int_0^\infty f(x,y)\, dy = 2e^{-2x}$$

d) Find marginal density function of y, $f_y(y)$

$$f_y(y) = \int_0^\infty f(x,y)\, dx = e^{-y}$$

d) Find the expected value and variance of X, $E(X)$ and $V(X)$

$$E(X) = \int_0^\infty \int_0^\infty x \cdot 2e^{-(2x+y)} dy\, dx = \frac{1}{2}$$

$$E(X^2) = \int_0^\infty \int_0^\infty x^2 \cdot 2e^{-(2x+y)} dy\, dx = \frac{1}{2}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

e) Find the expected value and variance of Y, $E(Y)$ and $V(Y)$

$$E(Y) = \int_0^\infty \int_0^\infty y \cdot 2e^{-(2x+y)} dy\, dx = 1$$

$$E(Y^2) = \int_0^\infty \int_0^\infty y^2 \cdot 2e^{-(2x+y)} dy\, dx = 2$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 2 - 1^2 = 1$$

f) Find the covariance of X and Y, $\text{Cov}(X, Y)$

$$E(XY) = \int_0^\infty \int_0^\infty xy \cdot 2e^{-(2x+y)} dy\, dx = \frac{1}{2}$$

$$Cov(x,y) = E(XY) - E(X)E(Y) = \frac{1}{2} - 1 \cdot \frac{1}{2} = 0$$

This is to be expected because X and Y are independent.

g) Find the conditional density $f_{x|y}(x)$ of X given Y

$$f_{x|y}(x) = \frac{f(x,y)}{f_y(y)} = \frac{2e^{-2x-y}}{e^{-y}}$$

where y is a contant equal to the given value of the random variable Y.

h) Find the conditional density $f_{y|x}(y)$ of Y given X

$$f_{y|x}(y) = \frac{f(x, y)}{f_x(x)} = \frac{2e^{-2x-y}}{2e^{-2x}}$$

where x is a contant equal to the given value of the random variable X.

3) A random vector $Y = \{Y_1, Y_2, \ldots, Y_n\}$ consists of n independent, identically distributed random variables Y_i, where each Y_i has a beta distribution,

$$f(y_i) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y_i^{\alpha-1}(1-y_i)^{\beta-1}, \quad y_i \in (0,1), \quad i = 1, \ldots, n$$

and α and β are positive constants common to all y_i.

a) Find the joint density function of the random vector Y. Because the Y_i are independent, the joint density is the product of the marginals, that is,

$$f(y_1, \ldots, y_n) = \prod_{i=1}^{n} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_i^{\alpha}(1-y_i)^{\beta}$$

or

$$f(y_1, \ldots, y_n) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^n \left(\prod_{i=1}^{n} y_i\right)^{\alpha} \left(\prod_{i=1}^{n}(1-y_i)\right)^{\beta}$$

b) Suppose $\overline{y}$ is the mean of the Y_i,

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

Use Theorem 5.12 to find the mean and variance of $\overline{y}$. (hint: $a_1 = a_2 = \cdots = a_n = 1/n$)

The expected value of $\overline{y}$ is:

$$E(\overline{Y}) = \sum_{i=1}^{n} \frac{1}{n}\mu = \mu = \frac{\alpha}{\alpha+\beta}$$

$$V(\overline{Y}) = \sum_{i=1}^{n} \frac{1}{n^2}\sigma^2 = \frac{\sigma^2}{n} = \frac{\alpha\beta}{n \cdot (\alpha+\beta)^2(\alpha+\beta+1)}$$