Margin of Error for Proportions

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Margin of Error

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Definition: The **margin of error** for an estimate $\hat{p}$ of a population proportion is defined to be $1/2$ the width of the **widest possible** 95% confidence interval for a sample size of n.

The margin of error is usually quoted as a percentage.

Margin of Error

Suppose a series of n independent Bernoulli trials with unknown probability of success p produces k successes.

As we have seen, an approximate 95% confidence interval for p is given by:

$$\left(\frac{k}{n} - 1.96\sqrt{\frac{\frac{k}{n}(1-\frac{k}{n})}{n}}, \quad \frac{k}{n} + 1.96\sqrt{\frac{\frac{k}{n}(1-\frac{k}{n})}{n}}\right)$$

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If $[L, U]$ is the 95% confidence interval for p, half the width of the interval is:

$$\frac{U - L}{2} = \left(\frac{k}{n} + 1.96\sqrt{\frac{\frac{k}{n}(1 - \frac{k}{n})}{n}}\right) - \left(\frac{k}{n} - 1.96\sqrt{\frac{\frac{k}{n}(1 - \frac{k}{n})}{n}}\right)$$

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Collecting terms and writing $\hat{p}$ for k/n, this becomes

$$\frac{U-L}{2} = 3.92\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

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$$\hat{p}(1-\hat{p}) \quad \text{is maximized when} \quad \hat{p} = \frac{1}{2}$$

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Taking the derivative with respect to $\hat{p}$ gives

$$\frac{d}{d\hat{p}}\left(\hat{p}(1-\hat{p})\right) \;=\; \frac{d}{d\hat{p}}\left(\hat{p}-\hat{p}^2\right) \;=\; 1-2\hat{p}$$

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$$\frac{d}{d\hat{p}}\left(\hat{p}(1 - \hat{p})\right) \;=\; \frac{d}{d\hat{p}}\left(\hat{p} - \hat{p}^2\right) \;=\; 1 - 2\hat{p}$$

It is clear that the derivative is zero when $\hat{p} = 1/2$ and that this represents a maximum because the second derivative is $-2 < 0$.

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We can obtain the widest possible 95% confidence interval for p by substituting $1/4$ for $\hat{p}(1-\hat{p})$:

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The margin of error is half the width of the widest possible 95% confidence interval, or

$$\frac{1}{2}\left(3.92\sqrt{\frac{1}{4n}}\right) \;=\; \frac{1.96}{2\sqrt{n}}$$

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Actually the only information we need is that $n = 1000$. Then

$$\text{margin of error} \ = \ \frac{1.96}{2\sqrt{1000}} \ = \ .03099$$

The margin of error is 3.1%.

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Using the fact that $n = 450$,

$$\text{margin of error} = \frac{1.96}{2\sqrt{450}} = .0462$$

The margin of error is 4.6%.