1) Suppose $Y = (Y_1, Y_2, Y_3, Y_4)$ is a vector of four independent, identically distributed random variables each with mean μ and variance σ^2. For each of the following estimators of μ, find the mean, variance, bias, and mean square error.

a)

$$\hat{Y}_1 = \frac{Y_1 + 3Y_3}{4}$$

Let

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} \quad t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \quad \mu_y = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \mu \\ \mu \end{bmatrix}$$

$$V = \sigma^2 I = \begin{bmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \sigma^2 & \\ & & & \sigma^2 \end{bmatrix}$$

Then:

$$E(\hat{Y}_1) = t'\mu_y = \left(\frac{1}{4}\right)[1\ 0\ 3\ 0]\begin{bmatrix} \mu \\ \mu \\ \mu \\ \mu \end{bmatrix} = \frac{4\mu}{4} = \mu$$

which can be written as

$$\sigma^2\left(\frac{1}{4}\right)^2[1\ 0\ 3\ 0]\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \frac{\sigma^2}{16}(1^2 + 0^2 + 3^2 + 0^2) = \frac{5\sigma^2}{8}$$

Since $E(\hat{Y}_1) = \mu$, the bias is zero and therefore the MSE is the same as $V(\hat{Y}_1)$.

b) This time

$$\hat{Y}_2 = \frac{Y_1 + 2Y_2 + Y_3}{4}$$

so

$$E(\hat{Y}_2) = t'\mu_y = \left(\frac{1}{4}\right)[1\ 2\ 1\ 0]\begin{bmatrix} \mu \\ \mu \\ \mu \\ \mu \end{bmatrix} = \frac{4\mu}{4} = \mu$$

and

$$\sigma^2\left(\frac{1}{4}\right)^2[1\ 2\ 1\ 0]\begin{bmatrix}1\\2\\1\\0\end{bmatrix}=\frac{\sigma^2}{16}(1^2+2^2+1^2+0^2)=\frac{3\sigma^2}{8}$$

Again $E(\hat{Y}_2)=\mu$ so the bias is zero and therefore the MSE is the same as $V(\hat{Y}_2)$.

c)

$$\hat{Y}_3=\overline{Y}=\frac{Y_1+Y_2+Y_3+Y_4}{4}$$

so

$$E(\hat{Y}_3)=t'\mu_y=\left(\frac{1}{4}\right)[1\ 1\ 1\ 1]\begin{bmatrix}\mu\\\mu\\\mu\\\mu\end{bmatrix}=\frac{4\mu}{4}=\mu$$

and

$$V(\hat{Y}_3)=\sigma^2\left(\frac{1}{4}\right)^2[1\ 1\ 1\ 1]\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\frac{\sigma^2}{16}(1^2+1^2+1^2+1^2)=\frac{\sigma^2}{4}$$

$E(\hat{Y}_2)=\mu$ so the bias is zero and again the MSE is the same as $V(\hat{Y}_2)$.

d)

$$\hat{Y}_4=\frac{Y_1-Y_2+Y_3-Y_4}{4}$$

so this time

$$E(\hat{Y}_4)=t'\mu_y=\left(\frac{1}{4}\right)[1\quad -1\quad 1\quad -1]\begin{bmatrix}\mu\\\mu\\\mu\\\mu\end{bmatrix}=0$$

and

$$V(\hat{Y}_4)=\sigma^2\left(\frac{1}{4}\right)^2[1\ -1\ 1\ -1]\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}=\frac{\sigma^2}{16}(1^2+(-1)^2+1^2+(-1)^2)=\frac{\sigma^2}{4}$$

The bias is

$$E(\hat{Y}_2) - \mu = -\mu$$

and the MSE is

$$\text{MSE}(\hat{Y}_1) = V(\hat{Y}_1) + (\text{Bias})^2 = \frac{\sigma^2}{4} + \mu^2$$

2) Suppose $(Y_1, \ldots, Y_n)$ is a random sample size n from a population with known mean μ. If $\hat{\theta}_2$ is an unbiased estimate of $E(Y^2)$ and $\hat{\theta}_3$ is an unbiased estimate of $E(Y^3)$, find an unbiased estimate of the third central moment of the underlying distribution.

The phrase *random sample* implies that the individual Y_i are independent and identically distributed (IID), and their common distribution is the "underlying distribution" that we are trying to find the third central moment of.

Let Y be a univariate random variable having the same distribution as each of the Y_i (this notation is a bit confusing because we have used Y to represent the random vector $(Y_1, \ldots, Y_n)$ elsewhere).

We are given two estimators $\hat{\theta}_2$ and $\hat{\theta}_3$ with the property that

$$E(\hat{\theta}_2) = E(Y^2) \quad \text{and} \quad E(\hat{Y}_3) = E(Y^3)$$

By definition, the third central moment of Y is

$$\mu_3 = E((Y - \mu)^3) = E(Y^3 - 3\mu Y^2 + 3\mu^2 Y - \mu^3)$$

$$= E(Y^3) - 3\mu E(Y^2) + 3\mu^2 E(Y) - \mu^3$$

$$= E(Y^3) - 3\mu E(Y^2) + 2\mu^3$$

The assumption is that we know μ, so we can treat it as a constant. Define

$$\hat{\mu_3} = \hat{\theta}_3 + 3\mu\hat{\theta}_2 + 2\mu^3$$

so

$$E(\hat{\mu_3}) = E(\hat{\theta}_3) + 3\mu E(\hat{\theta}_2) + 2\mu^3 = E(Y^3) + 3\mu E(Y^2) + 2\mu^3 = \mu_3$$

3) Suppose $(Y_1, \ldots, Y_n)$ is a random sample size n from a population with density function

$$f(y) = \alpha \cdot \frac{y^{\alpha-1}}{\theta^\alpha} \quad 0 \le y \le \theta$$

Let $\hat{\theta} = \max(Y_1, \ldots, Y_n)$ be an estimator of θ.

a) Find the density function of $\hat{\theta}$

The density function of the k^{th} order statistic is given by the formula:

$$g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k} \cdot f(y), \quad 0 \le y \le \theta$$

where

$$F(y) = \int_0^y f(t)dt \quad = \quad \int_0^y \alpha \cdot \frac{t^{\alpha-1}}{\theta^\alpha} dt = \frac{y^\alpha}{\theta^\alpha}, \quad 0 \le y \le \theta$$

The maximum is the order statistic with $k = n$, so on substitution we have

$$g_{(n)}(y) = \frac{n!}{(n-1)!(n-n)!} \left[\frac{y^\alpha}{\theta^\alpha}\right]^{n-1} \left[1 - \frac{y^\alpha}{\theta^\alpha}\right]^0 \cdot \frac{\alpha y^{\alpha-1}}{\theta^\alpha}, \quad 0 \le y \le \theta$$

which on simpification becomes

$$g_{(n)}(y) = \frac{n\alpha y^{n\alpha-1}}{\theta^{n\alpha}}, \quad 0 \le y \le \theta$$

Identifying $\hat{\theta}$ with y, we can rewrite this as a density function for $\hat{\theta}$:

$$f(\hat{\theta}) = \frac{n\alpha\hat{\theta}^{n\alpha-1}}{\theta^{n\alpha}}, \quad 0 \le \hat{\theta} \le \theta$$

b) Find the expected value, variance, and MSE of $\hat{\theta}$

This is an easy function to integrate, so we can directly evaluate the moments:

$$E(\hat{\theta}) = \int_0^\theta \hat{\theta} \cdot f(\hat{\theta})d\hat{\theta} = \int_0^\theta \hat{\theta} \cdot \frac{n\alpha\hat{\theta}^{n\alpha-1}}{\theta^{n\alpha}} d\hat{\theta} \quad = \quad \theta\left(\frac{n\alpha}{n\alpha+1}\right)$$

$$E(\hat{\theta}^2) = \int_0^\theta \hat{\theta}^2 \cdot f(\hat{\theta})d\hat{\theta} = \int_0^\theta \hat{\theta}^2 \cdot \frac{n\alpha\hat{\theta}^{n\alpha-1}}{\theta^{n\alpha}} d\hat{\theta} \quad = \quad \theta^2\left(\frac{n\alpha}{n\alpha+2}\right)$$

The bias is:

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta = \theta - \theta\left(\frac{n\alpha}{n\alpha+1}\right) = \frac{\theta}{n\alpha+1}$$

Note that as the sample size increases, that is, as $n \to \infty$, the bias tends to zero. An estimator with this property is said to be *asymptotically unbiased*.

The variance of $\hat{\theta}$ is

$$V(\hat{\theta}) = E(\hat{\theta}^2) - [E(\hat{\theta})]^2 = \frac{\theta^2}{n\alpha+2} - \left(\frac{n\alpha\theta}{n\alpha+1}\right)^2 = \frac{n\alpha \cdot \theta^2}{(n\alpha+2)(n\alpha+1)^2}$$

And finally the MSE of $\hat{\theta}$ is

$$\text{MSE}(\hat{\theta}) = V(\hat{\theta}) + \text{Bias}(\hat{\theta})^2$$

$$= \frac{\theta^2}{n\alpha+2} - \left(\frac{n\alpha\theta}{n\alpha+1}\right)^2 + \left(\frac{\theta}{n\alpha+1}\right)^2 = \frac{2\theta^2}{(n\alpha+1)(n\alpha+2)}$$

4) Suppose $(Y_1, \ldots, Y_9)$ is a sample of size $n = 9$ from a uniform distribution on $[0, 1]$. Let $\hat{\theta}$ be the sample median (i.e., the 5^{th} order statistic $Y_{(5)}$)

a) Show that $\hat{\theta}$ is an unbiased estimator for the population mean.

The density function of the k^{th} order statistic is given by the formula:

$$g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k} \cdot f(y), \quad 0 \le y \le \theta$$

where

$$f(y) = 1 \quad \text{and} \quad F(y) = \int_0^y f(t)dt \quad = \quad \int_0^y 1 \cdot dt = y, \quad 0 \le y \le 1$$

In this case $n = 9$ and $k = 5$, so on substitution we have

$$g_{(5)}(y) = \frac{9!}{(5-1)!(9-5)!}y^{5-1}(1-y)^{9-5} \cdot 1, \quad 0 \le y \le 1$$

which on simpification becomes

$$g_{(5)}(y) = \frac{9!}{4!4!}y^4(1-y)^4, \quad 0 \le y \le 1$$

and in terms of $\hat{\theta}$,

$$f(\hat{\theta}) = \frac{9!}{4!4!}\hat{\theta}^4(1-\hat{\theta})^4, \quad 0 \le \hat{\theta} \le 1$$

We recognize this as a beta distribution with parameters $\alpha = \beta = 5$, so the expected value is

$$E(\hat{\theta}) = \frac{\alpha}{\alpha + \beta} = \frac{1}{2}$$

so $\hat{\theta}$ is an unbiased estimator for the mean of the underlying population.

b) Find the variance and MSE of $\hat{\theta}$

From previous results (back cover of the text) we know

$$V(\hat{\theta}) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \theta + 1)} = \frac{25}{1100}$$

Because $\hat{\theta}$ is unbiased, the variance and the MSE are the same.

5) A very common problem is estimating the difference between the means of two populations. Suppose

$$Y_1 = (Y_{11}, Y_{12}, \ldots, Y_{1n_1})$$

is a random sample of size n_1 from a population with mean μ_1 and variance σ_1^2, and

$$Y_2 = (Y_{21}, Y_{22}, \ldots, Y_{2n_2})$$

is an independent random sample of size n_2 from a population with mean μ_2 and variance σ_2^2.

a) Show that $\hat{\theta} = \overline{Y}_1 - \overline{Y}_2$, the difference between the sample means, is an unbiased estimator for the difference between the two population means, $\mu_1 - \mu_2$.

We don't have the density function of the Y_i variables, but we can use the results for linear combinations of random variables that hold regardless of the distribution (as long as the means and variances are finite). In terms of the matrix formulation of Theorem 5.12, let Y be the $(n_1 + n_2) \times 1$ vector

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1n_1} \\ Y_{21} \\ \vdots \\ Y_{2n_2} \end{bmatrix}$$

The $(n_1 + n_2) \times 1$ vector of coefficients for the linear combination is:

$$t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 1/n_1 \\ \vdots \\ 1/n_1 \\ -1/n_2 \\ \vdots \\ -1/n_2 \end{bmatrix}$$

The $(n_1 + n_2) \times 1$ vector of means is:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_2 \end{bmatrix}$$

and finally the $(n_1 + n_2) \times (n_1 + n_2)$ (diagonal) variance-covariance matrix is:

$$V = \begin{bmatrix} \sigma_1^2 & & & & & \\ & \ddots & & & & \\ & & \sigma_1^2 & & & \\ & & & \sigma_2^2 & & \\ & & & & \ddots & \\ & & & & & \sigma_2^2 \end{bmatrix}$$

All of the covariances (the off-diagonal terms) are zero because we assume that the Y_{ij} are mutually independent.

Now using the matrix formulas,

$$E(t'Y) = t'\mu = \sum_1^{n_1} \frac{\mu_1}{n_1} + \sum_1^{n_2} -\frac{\mu_2}{n_2} = n_1 \cdot \frac{\mu_1}{n_1} - n_2 \cdot \frac{\mu_2}{n_2} = \mu_1 - \mu_2$$

b) Show that the variance of $\hat{\theta}$ is

$$V(\hat{\theta}) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

The variance of $\hat{\theta}$ is

$$V(\hat{\theta}) = V(t'Y) = t'Vt$$

We can write $t'V$ as the $1 \times (n_1 + n_2)$ vector:

$$t'V \quad = \quad \left[\frac{\sigma_1^2}{n_1} \cdots \frac{\sigma_1^2}{n_1} \;\; -\frac{\sigma_2^2}{n_2} \cdots -\frac{\sigma_2^2}{n_2}\right]$$

so

$$t'Vt = (t'V)t = \left[\frac{\sigma_1^2}{n_1} \cdots \frac{\sigma_1^2}{n_1} \;\; -\frac{\sigma_2^2}{n_2} \cdots -\frac{\sigma_2^2}{n_2}\right] \begin{bmatrix} 1/n_1 \\ \vdots \\ 1/n_1 \\ -1/n_2 \\ \vdots \\ -1/n_2 \end{bmatrix}$$

$$= \sum_1^{n_1} \left(\frac{\sigma_1^2}{n_1}\right)\left(\frac{1}{n_1}\right) + \sum_1^{n_2} \left(-\frac{\sigma_1^2}{n_2}\right)\left(-\frac{1}{n_2}\right)$$

$$= \quad n_1 \cdot \frac{\sigma_1^2}{n_1^2} + n_2 \cdot \frac{\sigma_2^2}{n_2^2} \quad = \quad \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$