Name:

1) Suppose $Y = (Y_1, Y_2, Y_3, Y_4)$ is a vector of four independent, identically distributed random variables each with mean μ and variance σ^2. For each of the following estimators of μ, find the mean, variance, bias, and mean square error.

a)

$$\hat{Y}_1 = \frac{Y_1 + 3Y_3}{4}$$

b)

$$\hat{Y}_2 = \frac{Y_1 + 2Y_2 + Y_3}{4}$$

c)

$$\hat{Y}_3 = \overline{Y} = \frac{Y_1 + Y_2 + Y_3 + Y_4}{4}$$

d)

$$\hat{Y}_4 = \frac{Y_1 - Y_2 + Y_3 - Y_4}{4}$$

2) Suppose $(Y_1, \ldots, Y_n)$ is a random sample size n from a population with known mean μ. If $\hat{\theta}_2$ is an unbiased estimate of $E(Y^2)$ and $\hat{\theta}_3$ is an unbiased estimate of $E(Y^3)$, find an unbiased estimate of the third central moment of the underlying distribution.

3) Suppose $(Y_1, \ldots, Y_n)$ is a random sample size n from a population with density function

$$f(y) = \alpha \cdot \frac{y^{\alpha-1}}{\theta^\alpha} \quad 0 \le y \le \theta$$

Let $\hat{\theta} = \max(Y_1, \ldots, Y_n)$ be an estimator of θ.

a) Find the density function of $\hat{\theta}$.

b) Find the expected value, variance, and MSE of $\hat{\theta}$

4) Suppose $(Y_1, \ldots, Y_9)$ is a sample of size $n = 9$ from a uniform distribution on $[0, 1]$. Let $\hat{\theta}$ be the sample median (i.e., the 5^{th} order statistic $Y_{(5)}$)

a) Show that $\hat{\theta}$ is an unbiased estimator for the population mean.

b) Find the variance and MSE of $\hat{\theta}$

5) A very common problem is estimating the difference between the means of two populations. Suppose

$$Y_1 = (Y_{11}, Y_{12}, \ldots, Y_{1n_1})$$

is a random sample of size n_1 from a population with mean μ_1 and variance σ_1^2, and

$$Y_2 = (Y_{21}, Y_{22}, \ldots, Y_{2n_2})$$

is an independent random sample of size n_2 from a population with mean μ_2 and variance σ_2^2.

a) Show that $\hat{\theta} = \overline{Y}_1 - \overline{Y}_2$, the difference between the sample means, is an unbiased estimator for the difference between the two population means, $\mu_1 - \mu_2$.

b) Show that the variance of $\hat{\theta}$ is

$$V(\hat{\theta}) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

(Hint: you can use Theorem 5.12, or the equivalent matrix formulation. Consider the $1 \times (n_1 + n_2)$ vector

$$Y' = (Y_1' \; Y_2') = (Y_{11}, Y_{12}, \ldots, Y_{1n_1}, Y_{21}, Y_{22}, \ldots, Y_{2n_2})$$

and the $1 \times (n_1 + n_2)$ vectors

$$t' = \left(\frac{1}{n_1}, \ldots, \frac{1}{n_1}, -\frac{1}{n_2}, \ldots, -\frac{1}{n_2}\right)$$

and

$$\mu' = (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2)$$

with V being a diagonal matrix of dimension $(n_1+n_2)\times(n_1+n_2)$ having its first n_1 diagonal elements equal to σ_1^2 and the next n_2 diagonal elements equal to σ_2^2. Then $\hat{\theta} = t'Y$, $E(\hat{\theta}) = t'\mu$, and $V(\hat{\theta}) = t'Vt$.)